# Notes: Change of Variables in One Dimension

Joshua Siktar

January 11, 2019

## 1 Context

These notes were created as a contribution for the Carnegie Mellon University Mathematics Department Spring 2019 Teaching Assistant Orientation.

## 2 Introduction

Oftentimes we are faced with integrating some function $f : \mathbb{R} \to \mathbb{R}$ either as an indefinite integral

$$\int f(x)dx$$

or as a definite integral

$$\int_a^b f(x)dx$$

Sometimes performing some type of substitution on the integrand helps with evaluating this integral in either case. This substitution will be of the form

$$u = g(x)$$

where $g$ is differentiable. In this case

$$\frac{du}{dx} = g'(x)$$

and

$$du = g'(x)dx$$

This technique can be used to rewrite the integral $\int f(x)dx$ as an integral over $u$.

## 3 Indefinite Integral I

**Example 1:** We compute $\int \cos x e^{\sin x} dx$.

**Solution:** Let $u = \sin x$. Then

$$\frac{du}{dx} = \cos x \Rightarrow du = \cos x dx$$

and substituting this into the integral yields

$$\int \cos x \cdot e^{\sin x} dx = \int \cos x \cdot e^{u} dx = \int e^{u} du = e^{u} + C$$

Note that as always we must undo any substitutions we make. Since $u = \sin x$,

$$\int \cos x e^{\sin x} dx = e^{\sin x} + C$$

## 4 Definite Integral I

**Example 2:** We compute $\int_0^{\pi} (4x + 7 + 3\cos x)\sin(2x^2 + 7x + 3\sin(x))dx$.

**Solution:** We first compute the indefinite integral $\int (4x+7+3\cos x)\sin(2x^2+7x+3\sin(x))dx$. Let $u = 2x^2 + 7x + 3\sin(x)$. Then

$$\frac{du}{dx} = 4x + 7 + 3\cos x \Rightarrow du = (4x + 7 + 3\cos(x))dx$$

so the integral becomes

$$\int \sin(u)du = -\cos(u) + C$$

Now undo the substitution:

$$\int (4x + 7 + 3\cos x)\sin(2x^2 + 7x + 3\sin(x))dx = -\cos(2x^2 + 7x + 3\sin(x))$$

We now compute the definite integral, acknowledging the bounds of integration:

$$\int_0^{\pi} (4x + 7 + 3\cos x)\sin(2x^2 + 7x + 3\sin(x))dx = \left[-\cos(2x^2 + 7x + 3\sin(x)\right]_0^{\pi} =$$

$$-\cos(2(\pi)^2+7(\pi)+3\sin(\pi))+\cos(2(0)^2+7(0)+3\sin(0)) = -\cos(2\pi^2+7\pi)+\cos(0) = 1-\cos(2\pi^2+7\pi)$$

## 5 Tips and Tricks

Some general advice for handling substitution techniques:

1. To identify a function $u = g(x)$ to use in a substitution, look for some function that appears in the integrand that also has its derivative appearing in the integrand.

2. Look to remove more intimidating expressions within the integrand: square roots, function compositions, etc.

3. Split an integral into a sum or difference of integrals.

4. Trial and error is important. Some u-substitutions actually make the integral in question once.

## 6 Indefinite Integral II

We now do a more complicated indefinite integral.

**Example 3:** We compute $\int \left(e^{e^x} + e^x\right) e^x dx$.

**Solution:** We expand the product:

$$\int \left(e^{e^x} + e^x\right) e^x dx = \int e^{e^x} e^x + e^{2x} dx = \int e^{e^x} e^x dx + \int e^{2x} dx$$

The second integral is easy:

$$\int e^{2x} dx = \frac{1}{2} e^{2x} + C_1$$

As for the first integral, let $u = e^x$, and then $du = e^x dx$, so

$$\int e^{e^x} e^x dx = \int e^{e^x} du = \int e^u du = e^u + C_2$$

We must undo this substitution:

$$\int e^{e^x} e^x dx = e^{e^x} + C_2$$

Finally combine the two integrals:

$$\int \left(e^{e^x} + e^x\right) e^x dx = \frac{1}{2} e^{2x} + e^{e^x} + C$$

where $C \in \mathbb{R}$ is arbitrary.