21269 Practice Midterm 1

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1 Problem 1

Let $E \subset \mathbb{R}^n$. If $f, g : E \to \mathbb{R}^n$ are Lipschitz continuous, prove that $f + g$ must also be Lipschitz continuous on E.

2 Problem 2

a. Let $E \subset \mathbb{R}^n$ and $\partial E = E$. Then show that $E' \subseteq \partial E$.

b. Prove the following claim or find a counterexample: let there be finitely many sets indexed E_i such that $\bigcup_{i=1}^n E_i \subset \mathbb{R}^n$. Then $\bigcap (E \setminus E_i')$ is an open set.

c. Let $E = \{x \in \mathbb{R}^{2016}.\ x_i > 0\ \forall i \in \{1, 2, ..., 2015\}, x_{2016} \in P\}$, where P is the set of prime numbers. Compute E°, E', $\bar{E}$, and ∂E, justifying your results (Hint: it may not be best to determine the sets in the order listed).

3 Problem 3

a. Compute $\lim_{x \to 0} \frac{1 - \cos^2 x}{x^2}$.

b. Show that $\lim_{x \to 0} \frac{1 - \cos^2 x}{x^3}$ without using L'Hopital's Rule.

c. Prove that $f(x) = \frac{1 - \cos^2 x}{x^3}$, where $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, is continuous.

4 Problem 4

For each proposed definition either state that it is correct as is, or modify the definition if it is incorrect.

a. Boundary point: $x \in E$ is a boundary point if $\exists \epsilon > 0.\ (B(x, \epsilon) \cap E \neq \emptyset) \wedge (B(x, \epsilon) \cap E^C \neq \emptyset)$.

b. Special orthogonal matrices $SO(n)$: $SO(n) = \{M \in \mathbb{R}^{n\times n}.\ (M \in O(n)) \wedge (\det(M) = 1)\}$.

c. Isolated point: $x \in E$ is an isolated point if $x \in (E')^C$.

d. Bounded sequence: If $\{x^{(k)}\}_{k=\ell}^{\infty} \subset \mathbb{R}^n$ is a sequence, it is bounded if $\exists M > 0.$ $|x^{(k)}| < M\ \forall k \geq \ell$.

e. Uniformly continuous on set $E \subset \mathbb{R}^n$: for some $x \in E$, $\forall \epsilon > 0$, $\exists \sigma > 0$ so that $((Y \in E) \wedge (|x - y| < \sigma)) \Rightarrow (|f(x) - f(y)| < \epsilon)$.

5 Problem 5

The supremum property of the real numbers has been a crucial base for many proofs throughout the course. Cite one theorem/lemma/corollary which utilized the supremum property of the real numbers, and explain why this property is instrumental in the proof (no need to give a full proof of the theorem you chose)

6 Problem 6

Consider this conjecture: if $z \in \mathbb{R}^n$, then we have $B_{||\cdot||_2}[z, r] \subset B_{||\cdot||_\infty}(z, r)$. If it is true prove it; if it is false, find an $x \in \mathbb{R}^n$ that invalidates the conjecture. Note that we let $||\cdot||_2$ be the 2-norm and $||\cdot||_\infty$ be the infinity norm.

7 Problem 7

Reprove this result from lecture: $\{x^{(k)}\}_{k=\ell}^{\infty}$ is convergent if and only if it is Cauchy.