21341 Practice Midterm 1

Joshua Siktar (jsiktar)

February 18, 2018

1 Problem 1

Let $u_1, u_2, ..., u_m$ and $v_1, ..., v_n$ be collections of vectors within a vector space V such that the vectors $u_1, u_2, ..., u_m$ are linearly independent and $v_1, ..., v_n$ are linearly independent. Prove that for any v_j ($j = 1, 2, ..., n$) there exists at most one vector u_i for which u_i and v_j are linearly dependent.

2 Problem 2

Let $Z_i\mathcal{P}(\mathbb{Z})$ denote the space of all polynomials with integer coefficients that are a multiple of i and underlying field $\mathbb{Z}$, where $i \in \mathbb{N}^+$. Let $Z_i\mathcal{P}^r(\mathbb{Z})$ denote the space of all polynomials of degree at most r having integer coefficients which are multiples of i and underlying field $\mathbb{Z}$.

Clearly all of these spaces are vector spaces.

a. Let $i \leq j$. Prove that $Z_i\mathcal{P}(\mathbb{Z})$ is a subspace of $Z_j\mathcal{P}(\mathbb{Z})$ if and only if i is a multiple of j.

b. Let $i, j, r \in \mathbb{N}^+$. Prove that $\dim(Z_i\mathcal{P}^r(\mathbb{Z})) = \dim(Z_j\mathcal{P}^r(\mathbb{Z}))$.

3 Problem 3

Let V and W be finite-dimensional vector spaces where $\dim(V) = k$ and $\dim(W) = n$, and let $T_1, T_2, ..., T_m \in \mathcal{L}(V, W)$.

a. Prove that

$$\text{range}[T_1] \cap ... \cap \text{range}[T_m]$$

is a subspace of W.

b. Prove that if

$$\dim(\text{range}[T_1] + ... + \text{range}[T_m]) = mn,$$

then $T_1, T_2, ..., T_m$ are all surjective.

4 Problem 4

Let V be a finite-dimensional vector space of dimension n and let V' be its dual space. Let A be the collection of all bases of V represented as ordered n-tuples of vectors, and let B be the collection of all bases of V' represented as ordered n-tuples of vectors. Fix some $v_1, ..., v_n$ basis of V and let $v_1^*, ..., v_n^*$ be its dual basis in V'. Let $\Pi : A \to B$ be defined so that for an arbitrary basis $w_1, ..., w_n$ of V,

$$\Pi((w_1, ..., w_n)) = (v_1^*(w_1), ..., v_n^*(w_n)).$$

Prove that $\Pi \in \mathcal{L}(A, B)$.

5 Problem 5

a. Let V a be finite dimensional vector space, let $T \in \mathcal{L}(V, W)$, and let $T' \in \mathcal{L}(W', V')$ be the dual map of T. Prove that

$$\dim(V') = \dim((\text{range}(T))^0) + \dim(((T))^0).$$

b. Under the same premises of the above part, prove

$$\dim(V') + \dim(\text{range}(T)) + \dim((T)) = 2\dim(V),$$