21341 Practice Midterm 1 Solutions

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1 Problem 1

Let $j \in \{1, 2, ..., n\}$. We will proceed by contradiction. Assume there exist u_i and u_k where $i < k$ and v_j is linearly dependent with u_i and with u_k. Then $\exists \lambda_i, \lambda_k, \mu_j, \rho_j \neq 0$ such that

$$\lambda_i u_i + \mu_j v_j = 0$$
$$\lambda_k u_k + \rho_j v_j = 0$$

Since $\rho_j \neq 0$, it has a unique inverse in the underlying field that we will call ρ_j^{-1}. Since $\lambda_k u_k + \rho_j v_j = 0$ it immediately follows that

$$\lambda_k \rho_j^{-1} \mu_j u_k + \mu_j v_j = 0$$

Subtracting this from $\lambda_i u_i + \mu_j v_j = 0$ gives

$$\lambda_i u_i - \lambda_k \rho_j^{-1} \mu_j u_k = 0$$

We can rewrite this equation in terms of all of the vectors $u_1, ..., u_m$:

$$0u_1 + ... + ou_{i-1} + \lambda_i u_i + 0u_{i+1} + ... + 0u_{k-1} - \lambda_k \rho_j^{-1} \mu_j u_k + 0u_{k+1} + ... + 0u_m = 0$$

Since $\lambda_i \neq 0$, we conclude that the vectors $u_1, u_2, ..., u_m$ are linearly dependent, a direct contradiction to their independence. □

2 Problem 2

We let $i \leq j$ and first assume i is a multiple of j. Then we show $Z_i\mathcal{P}(\mathbb{Z})$ is a subspace of $Z_j\mathcal{P}(\mathbb{Z})$. Since 0 is a multiple of any $Z_i\mathcal{P}(\mathbb{Z})$, the zero polynomial is in $Z_i\mathcal{P}(\mathbb{Z})$. Now we show closure under addition. Consider two polynomials in $Z_i\mathcal{P}(\mathbb{Z})$:

$$a_0 + a_1 x + ... + a_m x^m$$

$$b_0 + b_1 x + ... + b_n x^n$$

Assume without loss of generality that $n \geq m$. Then $a_0, a_1, ..., a_m, b_0, ..., b_n$ are all multiples of j. In particular, the sum of any of these coefficients is a multiple of j, so

$$(a_0 + b_0) + ... + (a_m + b_m)x^m + b_{m+1}x^{m+1} + ... + b_n x^n$$

is also in $Z_i\mathcal{P}(\mathbb{Z})$. Now let $\lambda \in \mathbb{Z}$. Since a_k is a multiple of j for any $k \in \{0, 1, ..., m\}$, so is λa_k. In particular,

$$\lambda a_0 + \lambda a_1 x + ... + \lambda a_m x^m \in Z_i\mathcal{P}(\mathbb{Z}),$$

and it follows that $Z_i\mathcal{P}(\mathbb{Z})$ is a subspace of $Z_j\mathcal{P}(\mathbb{Z})$.

Now we assume $Z_i\mathcal{P}(\mathbb{Z})$ is a subspace of $Z_j\mathcal{P}(\mathbb{Z})$ and want to show that i is a multiple of j. Assume for sake of contradiction that i is not a multiple of j. Then the constant polynomial i is not in $Z_j\mathcal{P}(\mathbb{Z})$, so $Z_i\mathcal{P}(\mathbb{Z})$ is not even contained by $Z_j\mathcal{P}(\mathbb{Z})$. $Z_i\mathcal{P}(\mathbb{Z})$ cannot possibly be a subspace of $Z_j\mathcal{P}(\mathbb{Z})$, a contradiction. $\square$

b. It suffices to construct a basis of $Z_i\mathcal{P}^r(\mathbb{Z})$ and a basis of $Z_j\mathcal{P}^r(\mathbb{Z})$ and show that they contain the same number of vectors. Let

$$B_i := \{i, ix, ix^2, ..., ix^r\}$$
$$B_j := \{j, jx, jx^2, ..., jx^r\}$$

Both of these collections of vectors have $r+1$ vectors, and we will show B_i is a basis of $Z_i\mathcal{P}^r(\mathbb{Z})$ and that B_j is a basis of $Z_j\mathcal{P}^r(\mathbb{Z})$. Clearly the collection $B_i := \{i, ix, ix^2, ..., ix^r\}$ is linearly independent since a polynomial is the zero polynomial only if all of its coefficients are zero. Moreover, these vectors span $Z_i\mathcal{P}^r(\mathbb{Z})$ because any polynomial in $Z_i\mathcal{P}^r(\mathbb{Z})$ can be written [uniquely] in the form

$$a_0 + a_1 x + ... + a_r x^r,$$

where $\exists \lambda_0, ..., \lambda_r \in \mathbb{Z}$ such that $a_k = \lambda_k i$ for each $k \in \{0, 1, ..., r\}$. Hence B_i is a basis of $Z_i\mathcal{P}^r(\mathbb{Z})$, and by identical reasoning, B_j is a basis of $Z_j\mathcal{P}^r(\mathbb{Z})$ as well. Since each basis contains $r+1$ vectors listed, we have $\dim(Z_i\mathcal{P}^r(\mathbb{Z})) = \dim(Z_j\mathcal{P}^r(\mathbb{Z}))$, as desired. $\square$

3 Problem 3

a. Let $S :=$ range$[T_1] \cap ... \cap$ range$[T_m]$. First note we proved in lecture that range$[T_i]$ is a subspace of W for each $i \in [m]$. Thus $0 \in$ range$[T_i]$ $\forall i \in [m]$, so in fact $0 \in S$. Now let $a, b \in S$ and $\lambda \in \mathbb{F}$, and we show that $a + b, \lambda a \in S$. Since each range$[T_i]$ is a subspace of W, in fact $a + b, \lambda a \in$ range$[T_i]$ $\forall i \in [m]$, so it follows that $a + b, \lambda a \in S$. Hence S is a subspace of W.

b. Since $\dim(W) = n$, and $\text{range}[T_i] \subset W \ \forall i \in [m]$, we have that $\dim(\text{range}[T_i]) \leq n \ \forall i \in [m]$, and then

$$\dim(\text{range}[T_1]) + ... + \dim(\text{range}[T_m]) \leq mn$$

Since W is finite dimensional, so is each $\text{range}[T_i]$, so by Exercise 2.C.14 from Homework 2,

$$\dim(\text{range}[T_1] + ... + \text{range}[T_m]) \leq \dim(\text{range}[T_1]) + ... + \dim(\text{range}[T_m])$$

As we are given $\dim(\text{range}[T_1] + ... + \text{range}[T_m]) = mn$, it follows that

$$\dim(\text{range}[T_1]) + ... + \dim(\text{range}[T_m]) = mn$$

Moreover, as the dimension of each subspace is at most n we have $\dim(\text{range}[T_1]) = n \ \forall i \in [m]$. In particular, it follows that $\text{range}[T_i] = W \ \forall i \in [m]$, meaning each map T_i is surjective, as desired. $\square$

4 Problem 4

Let $a, b \in A$, $\lambda \in \mathbb{F}$. Then we want to show

$$\Pi(a + b) = \Pi(a) + \Pi(b)$$
$$\Pi(\lambda a) = \lambda \Pi(a)$$

We will represent a and b coordinate-wise:

$$a := (x_1, ..., x_n)$$
$$b := (y_1, ..., y_n)$$

where $x_1, ..., x_n$ and $y_1, ..., y_n$ are bases of V. Then since the dual basis vectors $v_1^*, ..., v_n^*$ are linear maps from V to $\mathbb{F}$,

$$\Pi(a + b) = \Pi((x_1 + y_1, ..., x_n + y_n)) = (v_1^*(x_1 + y_1), ..., v_n^*(x_n + y_n)) =$$
$$(v_1^*(x_1) + v_1^*(y_1), ..., v_n^*(x_n) + v_n^*(y_n)) = (v_1^*(x_1), ..., v_n^*(x_n)) + (v_1^*(y_1), ..., v_n^*(y_n)) =$$
$$\Pi((x_1, ..., x_n)) + \Pi((y_1, ..., y_n)) = \Pi(a) + \Pi(b)$$

Furthermore,

$$\Pi(\lambda a) = \Pi((\lambda x_1, ..., \lambda x_n)) = (v_1^*(\lambda x_1), ..., v_n^*(\lambda x_n)) = (\lambda v_1^*(x_1), ..., \lambda v_n^*(x_n)) =$$
$$\lambda(v_1^*(x_1), ..., v_n^*(x_n)) = \lambda \Pi((x_1, ..., x_n)) = \lambda \Pi(a),$$

as desired. $\square$

5 Problem 5

a. Note that $V' = \mathcal{L}(V, \mathbb{F})$. Since V is finite dimensional, so is V', and so we know that

$$\dim(V') = \dim((T')) + \dim(\text{range}(T'))$$

Moreover, we proved in lecture that $(T') = (\text{range}(T))^0$ and $\text{range}(T') = ((T))^0$, so the result follows immediately.

b. Note that (T) and $\text{range}(T)$ are subspaces of V, so

$$\dim((T)) + \dim(((T))^0) = \dim(V)$$
$$\dim(\text{range}(T)) + \dim((\text{range}(T))^0) = \dim(V)$$

Substituting these into the result of part a gives

$$\dim(V') = \dim(V) - \dim(\text{range}(T)) + \dim(V) - \dim((T)) \Rightarrow$$
$$\dim(V') + \dim(\text{range}(T)) + \dim((T)) = 2\dim(V),$$

as desired. □