21369 Final Practice Problems

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May 6, 2018

1 Problem 1

Consider this variant on the RK-2 scheme:

$$\begin{cases} x(t+h) = x(t) + h\left(\frac{2}{9}K_1 + \frac{7}{9}K_2\right) \\ K_1 = f(t, x(t)) \\ K_2 = f\left(t + \frac{5}{9}h_1, x + \frac{5}{9}K_1\right) \end{cases}$$

Up to how many terms does this approximation agree with Taylor's Approximation for a fixed h?

2 Problem 2

Consider this variant on the RK-2 scheme:

$$\begin{cases} x(t+h) = x(t) + h\left(w_1 K_1 + w_2 K_2\right) \\ K_1 = f(t, x(t)) \\ K_2 = f\left(t + \alpha h_1, x + \beta K_1\right) \end{cases}$$

Find a solution to α, β, w_1, w_2 so that this scheme agrees with Taylor's Approximation up to the second order (i.e. with error term $O(h^3)$).

3 Problem 3

Consider the iterative matrix problem

$$x_{n+1} = (I - A^2)x_n + b$$

for which $A \in \mathbb{R}^{n\times n}$ has the following properties: $a_{ii} = 1\ \forall i \in [n]$. Moreover, $\forall i \in [n]$, $\exists! j \neq i$ for which $a_{ij} = 1$. For the fixed i and all $k \neq j$, $a_{ik} = 0$. Show that this method converges to the desired solution x.

4 Problem 4

Consider the iterative matrix problem

$$x_{n+1} = \begin{bmatrix} 0 & 1/6 & 0 & 0 & 0 \\ 0 & 0 & 1/6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} x_n + b$$

Clearly all of the eigenvalues are zero and so the matrix iterations converge to the desired solution. Provide another convergence proof revolving around bounding some induced norm of the matrix in the iteration above. Does the proof provided generalize to all Jordan block matrices?

5 Problem 5

Let A be a matrix with only positive eigenvalues. Let $\eta \in \mathbb{R}$ and suppose you are implementing the following matrix iteration problem on a computer for various values of $\omega \in \mathbb{R}$:

$$x^{k+1} = (I - \eta\omega A + A^2)x^k + c$$

However, being the careless programmer you are, you mistype the iterations to resemble this alternative iteration problem:

$$x^{k+1} = (I + \eta\omega A + A^2)x^k + c$$

Let's say ω was chosen so that the first method converges. Show that for any $\eta \in \mathbb{R}$, the typo cost you a valid convergence. That is, show that it is impossible for the second method to also converge with the a priori choice of ω.

6 Problem 6

Consider the iterative matrix problem

$$x^{k+1} = \begin{pmatrix} \lambda & 0 \\ 1 & 1 \end{pmatrix} x^k + c$$

Clearly we cannot prove convergence of this scheme through the eigenvalue method since one of the eigenvalues of the matrix is 1. We also investigated proving convergence by showing the matrix's norm is less than 1 with respect to some induced norm.

a. Show for any $\lambda \in \mathbb{R}$ we cannot obtain convergence of the method by letting the induced norm be $||\cdot||_1$.

b. Show for any $\lambda \in \mathbb{R}$ we cannot obtain convergence of the method by letting the induced norm be $||\cdot||_\infty$ (*Hint:* break into cases based on whether $|\lambda| \leq 1$ or $|\lambda| > 1$).

7 Problem 7

a. Choose coefficients A, B, and C for which the approximation scheme

$$\int_{-1}^{1} f dx \approx Af(-1) + Bf\left(\frac{1}{2}\right) + Cf(1)$$

is exact whenever f is a polynomial of degree at most 2.

b. Improve on the approximation scheme from part a by using Gaussian Quadrature to choose [three] new points on the interval $[-1, 1]$ to use in the approximation. Up to what degree polynomial is it exact?

8 Problem 8

a. Analyze the error of the Trapezoid Rule

$$\int_a^{a+h} f dx \sim \frac{f(a) + f(a+h)}{2} \cdot h$$

b. Analyze the error of the Midpoint Rule

$$\int_a^{a+h} f dx \sim h \cdot f\left(a + \frac{h}{2}\right)$$

c. Analyze the error of Simpson's Rule

$$\int_a^{a+2h} f dx \sim \frac{h}{3}[f(a) + 4f(a+h) + f(a+2h)]$$

9 Problem 9

Find coefficients α, β, γ such that the order of the error term for

$$\int_a^{a+h} f dx \sim \frac{h}{3}\left(\alpha f(a) + \beta f\left(a + \frac{h}{2}\right) + \eta f(a+h)\right)$$

is of high degree as possible. What is the leading term in the error?

10 Problem 10

Prove that $||\cdot|| : \mathbb{R}^{n\times n} \to \mathbb{R}$ is an induced norm on matrices if and only if $\alpha||\cdot|| : \mathbb{R}^{n\times n} \to \mathbb{R}$ is an induced norm on matrices only for $\alpha = 1$.

11 Problem 11

Consider the generic matrix iteration scheme

$$Qx^{n+1} + (A - Q)x^n = b$$

Using this scheme, justify why the condition $||I - Q^{-1}A|| < 1$ for some induced norm $||\cdot||$ is sufficient for [linear] convergence of the method (the derivation comes directly from the lecture notes).

12 Problem 12

a. Find a cubic polynomial p with initial data $p(0) = 3$, $p(1) = 7$, $p''(2) = 5$, and $p'(1) = 4$.

b. Find a quartic polynomial q sharing the same initial data as p but also requiring $q(2) = 23$.

13 Problem 13

a. Find coefficients $A, B \in \mathbb{R}$ which best approximate

$$f'(x) \approx Af\left(x - \frac{h}{3}\right) + Bf\left(x + \frac{2h}{3}\right)$$

b. Find the error of the scheme derived in part a.

c. Let $\phi(h) = Af\left(x - \frac{h}{3}\right) + Bf\left(x + \frac{2h}{3}\right)$ with A, B determined in part a. Using $\phi(h)$ and $\phi\left(\frac{h}{2}\right)$, perform one iteration of Richardson Extrapolation to improve the error of the scheme from part b by one power of h.