21369 Practice Midterm 1 Solutions

Joshua Siktar (jsiktar)

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1 Solution to Problem 1

a. The natural choice of polynomial is $f(x) = x^2 - 42$. Then $f'(x) = 2x$. Using Newton's Method twice times gives

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 6 - \frac{f(6)}{f'(6)} = 6 - \frac{6^2 - 42}{12} = \frac{13}{2}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = \frac{13}{2} - \frac{\left(\frac{13}{2}\right)^2 - 42}{2 \cdot \frac{13}{2}} = \frac{167}{26}$$

b. We will demonstrate using $f(x)$ from part a, though answers will vary if you chose a different f. $g(x) = (x^2 - 42)^2$, which notably has $\sqrt{42}$ as a double root. Then

$$g'(x) = 4x(x^2 - 42)$$

by the Chain Rule, which has $\sqrt{42}$ as a root itself. In other words, Newton's Method is not effective because the root we are trying to approximate is not simple.

2 Solution to Problem 2

We use Taylor's Theorem on each term in the proposed approximation:

$$f(x + h^2) = f(x) + h^2 f'(x) + \frac{h^4}{2} f''(x) + \frac{h^6}{6} f'''(\xi_1)$$

$$f(x - h^2) = f(x) - h^2 f'(x) + \frac{h^4}{2} f''(x) - \frac{h^6}{6} f'''(\xi_2)$$

$$f(x + h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{6} f'''(\xi_3)$$

$$f(x - h) = f(x) - hf'(x) + \frac{h^2}{2} f''(x) - \frac{h^3}{6} f'''(\xi_4),$$

where ξ_1 is between x and $x+h^2$, and similarly for the other ξ_i. Combine the first two equations and the last two equations, and multiply by the respective constants A and B:

$$Af(x+h^2) - Af(x-h^2) = 2Ah^2 f'(x) + \frac{Ah^6}{3}[f'''(\xi_1) + f'''(\xi_2)]$$

$$Bf(x+h) - Bf(x-h) = 2hBf'(x) + \frac{Bh^3}{3}[f'''(\xi_3) + f'''(\xi_4)]$$

Then

$$Af(x+h^2) + Bf(x+h) - Bf(x-h) - Af(x-h^2) =$$

$$2Ah^2 f'(x) + 2hBf'(x) + \frac{Ah^6}{3}[f'''(\xi_1) + f'''(\xi_2)] + \frac{Bh^3}{3}[f'''(\xi_3) + f'''(\xi_4)]$$

The desired approximation scheme is impossible as there is no $f''(x)$ term. We cannot solve equations to identify A and B and obtain anything meaningful. $\square$

3 Solution to Problem 3

As $x_k = r + e_k$ for any $k \in \mathbb{N}^+$, we can use Taylor's Theorem on f and f' up to the f'' term:

$$f(x_n) \approx f(r) + e_n f'(r) + \frac{e_n^2}{2} f''(r) = e_n f'(r) + \frac{e_n^2}{2} f''(r)$$

$$f'(x_n) \approx f'(r) + e_n f''(r)$$

$$f(x_{n-1}) \approx f(r) + e_{n-1} f'(r) + \frac{e_{n-1}^2}{2} f''(r) = e_{n-1} f'(r) + \frac{e_{n-1}^2}{2} f''(r)$$

$$f'(x_{n-1}) \approx f'(r) + e_{n-1} f''(r)$$

Note that we can rewrite the iteration scheme with e_k terms:

$$x_{n+1} = \frac{2}{3}\left(x_n + \frac{f(x_n)}{f'(x_n)}\right) + \frac{1}{3}\left(x_{n-1} + \frac{f(x_{n-1})}{f'(x_{n-1})}\right) \Rightarrow$$

$$r + e_{n+1} = \frac{2}{3}\left(r + e_n + \frac{f(x_n)}{f'(x_n)}\right) + \frac{1}{3}\left(r + e_{n-1} + \frac{f(x_{n-1})}{f'(x_{n-1})}\right) \Rightarrow$$

$$e_{n+1} = \frac{2}{3}\left(e_n + \frac{f(x_n)}{f'(x_n)}\right) + \frac{1}{3}\left(e_{n-1} + \frac{f(x_{n-1})}{f'(x_{n-1})}\right)$$

Now we can estimate $\left|e_n + \frac{f(x_n)}{f'(x_n)}\right|$ as follows:

$$\left|e_n + \frac{f(x_n)}{f'(x_n)}\right| \approx \left|e_n + \frac{e_n f'(r) + \frac{e_n^2}{2} f''(r)}{f'(r) + e_n f''(r)}\right| \approx \left|\frac{2e_n f'(r) + \frac{3e_n^2}{2} f''(r)}{f'(r) + e_n f''(r)}\right|$$

Since $f''(x) \geq 0$ on $\mathbb{R}$, we obtain

$$\left|\frac{2e_n f'(r) + \frac{3e_n^2}{2} f''(r)}{f'(r) + e_n f''(r)}\right| \leq \left|\frac{2e_n f'(r) + 2e_n^2 f''(r)}{f'(r) + e_n f''(r)}\right| = 2|e_n|$$

We can also perform the same process on $e_{n-1} + \frac{f(x_{n-1})}{f'(x_{n-1})}$ to get the analogous estimate

$$\left|e_{n-1} + \frac{f(x_{n-1})}{f'(x_{n-1})}\right| \leq 2|e_{n-1}|$$

Now we use our two estimates in conjunction with the Triangle Inequality:

$$|e_{n+1}| = \left|\frac{2}{3}\left(e_n + \frac{f(x_n)}{f'(x_n)}\right) + \frac{1}{3}\left(e_{n-1} + \frac{f(x_{n-1})}{f'(x_{n-1})}\right)\right| \leq$$
$$\frac{2}{3}\left|e_n + \frac{f(x_n)}{f'(x_n)}\right| + \frac{1}{3}\left|e_{n-1} + \frac{f(x_{n-1})}{f'(x_{n-1})}\right| \leq \frac{4}{3}|e_n| + \frac{2}{3}|e_{n-1}|,$$

as desired. □

Remark: This bound is not good enough to assure convergence because of the factor of $\frac{4}{3}$. The student has more thinking to do!