21369 Practice Midterm 2

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1 Problem 1

a. Let f be a Riemann Integrable function, and say we seek to approximate

$$\int_0^1 f(x)dx \approx Af\left(\frac{1}{2}\right) + B\left(\frac{1}{4}\right) + Cf(0)$$

for some constants $A, B, C \in \mathbb{R}$. Find the $A, B, C \in \mathbb{R}$ such that the above approximation is exact for polynomials up to degree 2.

b. Generalize your solution to part a to some arbitrary interval $[a, b] \subseteq \mathbb{R}$, where $a < b$.

c. Let $n > 2$ be arbitrary. Using the formula from part a, compute the error of our approximation scheme for computing $\int_0^1 x^n dx$.

2 Problem 2

Let $E \subset \mathbb{R}$ be a set. A **lower bound** of the set E is some $b \in \mathbb{R}$ such that $b \leq x \; \forall x \in E$. The infimum of the set E, when it exists, is denoted $\inf(E)$. This value is a lower bound t of E such that for any lower bound b of E, $b \leq t$. In this problem we fix some matrix $A \in \mathbb{R}^{n \times n}$ and define the 2γ-norm as a function $||\cdot||_{2\gamma} : \mathbb{R}^n \to \mathbb{R}$ to be

$$||Ax||_{2\gamma} = \inf_{n \in \mathbb{N}^+} ||A^n x||_2,$$

where $||\cdot||_2$ is the 2-norm on vectors in $\mathbb{R}^n$. [sidenote: the notion of a 2γ-norm is one I devised myself for use on this practice exam.]

a. Assume E is a set for which an infimum exists. Prove that the infimum is unique (because we want to implicitly assume this in later parts of the problem).

b. Suppose that A is a nilpotent matrix, for which $\exists k \in \mathbb{N}^+$ such that $A^k = 0$. Prove that for such A, the 2γ-norm is *not* a norm.

c. Contrary to part b, assume now that A is not a nilpotent matrix, that is, $\forall n \in \mathbb{N}^+$, $A^n \neq 0$. Prove that for such A, the 2γ-norm *is* a norm. (Hint: you may use the following two facts without proof: let $a \geq 0$ and for a set E with an infimum define $aE = \{ax : x \in E\}$. Then $\inf_{x\in E} aE = a \cdot \inf_{x\in E} E$. Also, for any $x, y \in \mathbb{R}^n$ you may use that $\inf_{n\in\mathbb{N}^+}(||A^n x||_2 + ||A^n x||_2) \leq \inf_{n\in\mathbb{N}^+}||A^n x||_2 + \inf_{n\in\mathbb{N}^+}||A^n y||_2)$.

3 Problem 3

Consider the ODE

$$\begin{cases} x'(t) = 5e^t + x + t \\ x(0) = 1 \end{cases}$$

Perhaps we want to approximate a solution to this ODE.

a. Let $h > 0$ be some arbitrary step size, where we "begin" approximating a solution to this ODE at $t = 0$. Using the left-hand approximation rule, compute $x(h)$.

b. Now recall the Runge-Katta Scheme of Order 2:

$$x(t+h) = x(t) + \frac{h}{2} f(t,x) + \frac{h}{2} f(t+h, x + hf(t,x))$$

Find an alternative approximation for the ODE using this scheme, expanding the $f(t+h, x+hf(t,x))$ term with 2-dimensional Taylor's Formula up to the 1st order partial derivatives and simplifying. That is, find another expression for $x(h)$.

4 Problem 4

a. If solving $Ax = b$ where A is a diagonal matrix without zero as an eigenvalue, show that the Gauss-Seiden method for estimating the solution x indeed converges to the solution.

b. Again consider estimating a solution to $Ax = b$. The Gauss-Seiden method uses an "initial matrix" $Q = D + L$, where D and L are the diagonal and lower triangle parts of A, respectively. Explain why the analogous algorithm with starting matrix $Q = D + U$, where U is the upper triangle part of A, will also converge, under analogous conditions to those guaranteeing the convergence of Gauss-Seiden (by no means am I expecting a formal proof here!).