21369 Practice Midterm 2 Solutions

Joshua Siktar (jsiktar)

March 27, 2018

1 Solution to Problem 1

a. We make the approximation exact by letting $f(x) = 1, x, x^2$ in the approximation scheme and generating a system to solve for A, B, C:

$$\int_0^1 1dx = A + B + C = 1$$

$$\int_0^1 xdx = \frac{A}{2} + \frac{B}{4} = \frac{1}{2}$$

$$\int_0^1 x^2dx = \frac{A}{4} + \frac{B}{16} = \frac{1}{3}$$

One can solve this system of linear equations, ideally by first solving the last two equations as just a system of A and B. The unique result is $(A, B, C) = \left(\frac{5}{3}, -\frac{4}{3}, \frac{2}{3}\right)$. Note that since this approximation is exact for $f(x) = 1, x, x^2$, it is exact for any polynomial of degree ≤ 2 because any such polynomial is a linear combination of $1, x, x^2$.

b. The change of variables formula to convert an approximation from an interval $[c, d]$ to an interval $[a, b]$ is

$$\lambda(t) = \left(\frac{b-a}{d-c}\right)t + \frac{ad-bc}{d-c}$$

Since $c = 0$ and $d = 1$, in our case,

$$\lambda(t) = (b-a)t + a$$

Now we perform a change of variables, letting $x = \lambda(t)$, and then $\frac{dx}{dt} = b - a$, so

$$\int_a^b f(x)dx = (b-a)\int_0^1 f((b-a)t + a)dt$$

Using our approximation scheme from part a, this means that

$$\int_a^b f(x)dx \approx \frac{5}{3}(b-a)f\left(\frac{b-a}{2}+a\right) - \frac{4}{3}(b-a)f\left(\frac{b-a}{4}+a\right) + \frac{2}{3}(b-a)f(a)$$

c. Let $n > 2$ be arbitrary. Then

$$\int_0^1 x^n dx = \left[\frac{1}{n+1}x^{n+1}\right]_0^1 = \frac{1}{n+1}$$

Our approximation scheme for $f(x) = x^n$ gives

$$\frac{5}{3}\left(\frac{1}{2}\right)^n - \frac{4}{3}\left(\frac{1}{4}\right)^n + \frac{2}{3}f(0) = \frac{5}{3 \cdot 2^n} - \frac{1}{3 \cdot 4^{n-1}}$$

As a result, the error term is

$$\left|\frac{1}{n+1} - \frac{5}{3 \cdot 2^n} + \frac{1}{3 \cdot 4^{n-1}}\right|.$$

2 Solution to Problem 2

a. Let E be a set with two infima a and b. Then by the definition of infima, for any s and t that are lower bounds of E (which exist since every infima is a lower bound), $s \leq a$ and $t \leq b$. As a and b are lower bounds, set $s = b$ and $t = a$, so we get $b \leq a$ and $a \leq b$, or $a = b$. Hence the infimum is unique.

b. One of the conditions for a function being a norm is that it only outputs zero if the argument is zero. Thus it suffices to show that $\exists x \in \mathbb{R}^n \setminus \{0\}$ for which $||Ax||_{2\gamma} = 0$. Let $x = e_1$, where e_1 is the "first" standard basis vector on $\mathbb{R}^n$, i.e. $e_1 = (1, 0, 0, ..., 0)$. Since A is nilpotent, $\exists k \in \mathbb{N}^+$ for which $A^k = 0$. Clearly for any $m > k$, $A^m = 0$ as well since $A^m = A^k A^{m-k}$. Then $A^m e_1 = 0 \ \forall m \geq k$. Since $|| \cdot ||_2$ is a norm, this means $||A^m e_1||_2 = 0 \ \forall m \geq k$. Moreover, $||A^n e_1||_2 \geq 0 \ \forall n \in \mathbb{N}^+$, so by the definition of an infimum, $\inf_{n \in \mathbb{N}^+} ||A^n e_1||_2 = 0$. This means that $||Ae_1||_{2\gamma} = 0$, so $|| \cdot ||_{2\gamma}$ is not a norm if A is nilpotent.

c. We have four conditions to prove to assure $|| \cdot ||_{2\gamma}$ is a norm. In doing so, assume $\lambda \in \mathbb{R}$, $x, y \in \mathbb{R}^n$, and implicitly use the fact that $|| \cdot ||_2$ is a norm wherever necessary.

i) We first show that $||Ax||_{2\gamma} = 0$ if $x = 0$.

$$||A(0)||_{2\gamma} = \inf_{n \in \mathbb{N}^+} ||A^n(0)||_2 = \inf_{n \in \mathbb{N}^+} 0 = 0$$

ii) Now assume that $||Ax||_{2\gamma} = 0$ for some $x \in \mathbb{R}^n$ and we show that $x = 0$. We have that

$$||Ax||_{2\gamma} = \inf_{n \in \mathbb{N}^+} ||A^n x||_2 = 0$$

Since $|| \cdot ||_2$ is a function that maps only onto nonnegative values, and the infimum above is zero, we have that $||A^n x||_2 = 0 \; \forall n \in \mathbb{N}^+$, and $A^n x = 0$. At the same time, $A^n \neq 0 \; \forall n \in \mathbb{N}^+$. The only possibility is that $x = 0$.

iii) We show that $||\lambda Ax||_{2\gamma} = |\lambda| \cdot ||Ax||_{2\gamma}$, using the properties of the 2-norm and the hint provided:

$$||\lambda Ax||_{2\gamma} = ||A(\lambda x)||_{2\gamma} = \inf_{n \in \mathbb{N}^+} ||A^n(\lambda x)||_{2\gamma} = \inf_{n \in \mathbb{N}^+} |\lambda| ||A^n x|| = |\lambda| \inf_{n \in \mathbb{N}^+} ||A^n x|| = |\lambda| \cdot ||Ax||_{2\gamma}$$

iv) Finally we show that $||A(x+y)||_{2\gamma} \leq ||Ax||_{2\gamma} + ||Ay||_{2\gamma}$:

$$||A(x+y)||_{2\gamma} = \inf_{n \in \mathbb{N}^+} ||A^n x + A^n y||_2$$

Note that we can use the Triangle Inequality on $||\cdot||_2$ to get $||A^n x + A^n y||_2 \leq ||A^n x||_2 + ||A^n y||_2$ for each $n \in \mathbb{N}^+$. Essentially, the infimum over all $n \in \mathbb{N}^+$ cannot decrease since no expression $||A^n x||_2 + ||A^n y||_2$ can be smaller than $\inf_{n \in \mathbb{N}^+} ||A^n x + A^n y||_2$, so

$$\inf_{n \in \mathbb{N}^+} ||A^n x + A^n y||_2 \leq \inf_{n \in \mathbb{N}^+} (||A^n x||_2 + ||A^n x||_2) \leq \inf_{n \in \mathbb{N}^+} ||A^n x||_2 + \inf_{n \in \mathbb{N}^+} ||A^n y||_2 = ||Ax||_{2\gamma} + ||Ay||_{2\gamma},$$

as desired. □

3 Solution to Problem 3

a. The left-hand approximation rule for solving ODEs is

$$\int_t^{t+h} f(s, x(s)) ds \approx hf(t, x(t)).$$

For this problem, h is arbitrary, $t = 0$, and $f(t, x(t)) = 5e^t + x + t$, so the approximation scheme actually equals

$$x(t+h) - x(t) \approx hf(t, x(t)) \Rightarrow$$
$$x(h) - x(0) \approx h(5e^t + x(t) + t) \Rightarrow$$
$$x(h) - 1 \approx h(5 + 1) \Rightarrow$$
$$x(h) \approx 6h + 1$$

b. First, plug in $f(t,x)$:

$$x(t+h) = x(t) + \frac{h}{2}(5e^t + x + t) + \frac{h}{2} f(t+h, x + h(5e^t + x + t)) \Rightarrow$$

$$x(t+h) = x(t) + \frac{5he^t}{2} + \frac{hx}{2} + \frac{th}{2} + \frac{h}{2} f(t+h, x + 5he^t + hx + th)$$

We need to use two-dimensional Taylor on $f(t+h, x + (5he^t + hx + th))$. As we only need the first-order partial derivatives, we expand as follows:

$$f(t+h, x + (5he^t + hx + th)) \approx f(t,x) + h\frac{\partial f}{\partial t} + (5he^t + hx + th)\frac{\partial f}{\partial x}$$

Since we have an explicit formula for f, we compute the two partial derivatives explicitly:

$$\frac{\partial f}{\partial t} = 5e^t + x'(t) + 1 = 5e^t + 5e^t + x + t + 1 = 10e^t + x + t + 1$$

$$\frac{\partial f}{\partial x} = 1$$

As a result,

$$f(t+h, x + (5he^t + hx + th)) \approx 5e^t + x + t + h(10e^t + x + t + 1) + (5he^t + hx + th) \Rightarrow$$

$$f(t+h, x + (5he^t + hx + th)) \approx 5e^t + x + t + h + 15he^t + 2hx + 2th$$

Substituting this back into the expression for $x(t+h)$ yields

$$x(t+h) \approx x(t) + \frac{5he^t}{2} + \frac{hx}{2} + \frac{th}{2} + \frac{h}{2}(5e^t + x + t + h + 15he^t + 2hx + 2th) \Rightarrow$$

$$x(t+h) \approx x(t) + 5he^t + hx + th + \frac{h^2}{2} + \frac{15h^2e^t}{2} + h^2x + th^2$$

Since $t = 0$ and $x = 1$ for our ODE, this simplifies even further:

$$x(h) \approx 1 + 5he^0 + h(1) + (0)h + \frac{h^2}{2} + \frac{15h^2e^0}{2} + h^2(1) + 0h^2 \Rightarrow$$

$$x(h) \approx 1 + 6h + 9h^2$$

4 Solution to Problem 4

a. A is a matrix of the form

$$\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

Its eigenvalues are $\lambda_1, \lambda_2, \lambda_3$ (immediate). If they are all nonzero, then λ_i has a larger magnitude than the other values in row i for $i = 1, 2, 3$, so the matrix is diagonally dominant, and we proved in lecture that Gauss-Seiden converges whenever the matrix A is diagonally dominant. $\square$

b. We proved in lecture that the Gauss-Seiden Method converges to the solution $Ax = b$ if A is diagonally dominant. It is immediate by checking entries of the form a_{ij} of A that A is diagonally dominant if and only if A^T is diagonally dominant. The lower triangular part of A becomes the upper triangular part of A^T, so we can essentially translate the solving of $A^T x = b$ using $Q = U + D$ of A^T to using Gauss-Seiden on $Ax = b$.