21632 Final Practice Problems

Joshua Siktar

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1 Disclaimer

These problems do not necessarily reasonably represent the topics that could be on the actual exam. I pulled ideas from the lecture notes and textbook in design of these problems.

2 Breaking The Maximum Principle

a. Consider the piecewise function $u : \mathbb{R} \to \mathbb{R}$ such that

$$u(x) = \begin{cases} 2x, \ x \leq \frac{3}{2} \\ 3, \ x > \frac{3}{2} \end{cases}$$

Using the Maximum Principle (for Laplace's Equation) justify that u cannot be harmonic on all of $\mathbb{R}$.

b. Recall the strong maximum principle states that if the maximum of a harmonic function is attained on the interior of a connected set then the function must be constant. Propose an alternative argument if in fact the domain is convex.

3 Don't be me and Make Careless Errors

Below I copy my solution to Problem 4a from Homework 2, but there is an error that completely destroyed the validity of my proof. Can you find it?

There will either be zero solutions for the BVP or at least one. If there are zero solutions then we are done. If there is at least one, we will show it is unique.

Let $u, \widetilde{u}$ be solutions of the BVP and we will demonstrate $u = \widetilde{u}$. Let $w := u - \widetilde{u}$. Then $u_t = u_{xx}$ and $\widetilde{u_t} = \widetilde{u_{xx}}$, so clearly

$$w_t = w_{xx}$$

and

$$w_{xx} - w_t = 0 \Rightarrow$$

$$w_t(w_t - w_{xx}) = 0 \Rightarrow$$

$$0 = \int_0^1 w_t(w_t - w_{xx})dx = \int_0^1 w_t^2 - w_t w_{xx} dx$$

Since $w_t = -w_{xx}$, the above implies

$$0 = \int_0^1 w_t^2 dx + \int_0^1 w_{xx}^2 dx = \int_0^1 w_t^2 + w_{xx}^2 dx$$

The integrand is nonnegative so we must have $w_t^2 + w_{xx}^2 dx = 0$ for all $(x, t) \in U$. In particular it follows that $w_t = w_{xx} = 0$ on U.

With this in mind we let

$$f(t) := \int_0^1 w(x,t)^2 dx$$

on $t \in [0, T]$. In particular, the boundary conditions give

$$w(x, 0) = u(x, 0) - \widetilde{u}(x, 0) = g(x) - g(x) = 0$$

so $f(0) = 0$. Moreover, the integrand is smooth so we can differentiate under the integral sign:

$$f'(t) = \int_0^1 2ww_t dx$$

But we proved $w_t = 0$ on U, so $f'(t) = 0$. This means that $f(t) = 0\ \forall t \in [0, T]$. In particular, since w^2 is a nonnegative integrand being integrated over $(0, 1)$, we have that $w^2 = 0$ on U, so w is identically zero on U. From here we conclude $u = \hat{u}$, as desired. $\square$

4 Denying Compact Support to Initial Data

In utilizing Duhamel's Principle to solve the nonhomogeneous boundary value problem

$$\begin{cases} u_t - \triangle u = f,\ (x, t) \in \mathbb{R}^n \times \{t > 0\} \\ u = 0, (x, t) \in \mathbb{R}^n \times \{t = 0\} \end{cases}$$

we verified in Evans that the following works as a solution if we assume that $f \in C^{2,1}(\mathbb{R}^n \times [0, \infty)$ and compactly supported:

$$u(x,t) = \int_0^t \frac{1}{(4\pi(t-s))^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s)dyds$$

Consider the following proposed method to generalize this solution to when f may *not* be compactly supported (but still have the same differentiability). Show that f can be written as an infinite sum of compactly supported functions, and discuss whether interchanging the resulting infinite sum with the integrals above is justified.

5 An Infinity-Norm PDE

Let $U \subset \mathbb{R}^n$ be open and bounded with a C^1 boundary. Let $g \in C(\bar{U})$. Consider this boundary value problem

$$\begin{cases} \max_{1 \leq i \leq n} |(\nabla u(x))_i| = u(x)^4, \ x \in U \\ u(x) = g(x), \ x \in \partial U \end{cases}$$

Clearly the zero function is a nonnegative solution to the BVP. Show it is the only nonnegative solution.

6 Just an Exponential Away

Solve the following nonlinear PDE in $\mathbb{R} \times \mathbb{R}$:

$$\begin{cases} ww_{tt} - w_t^2 - ww_{xx} + w_x^2 = 0, \ x, t > 0 \\ w(x, 0) = \ln(x \sin(x) + 2), \ x > 0, \ t = 0 \\ w_t(x, 0) = \frac{x^2}{2} + \ln(x + 1), \ x > 0, \ t = 0 \end{cases}$$

Hint: use a smooth transformation to a linear PDE, see Evans 4.4.1.

7 The Cousin of Grönwall

Let $\phi \in C^1([0, \infty))$. Let $C, K > 0$ be constants such that

$$\phi(t) \leq C + \int_0^t K\phi(s)ds$$

for all $t \geq 0$. Also assume that $\phi'(t) \leq -K\phi(t)$ for all $t > 0$. Then show that for all $t > 0$, $\phi(t) \leq \phi(0)e^{Kt}$.

8 The Evils of Weak Solutions

Consider the conservation law PDE known as Burgers' Equation:

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0, \ (x,t) \in \mathbb{R} \times (0, \infty) \\ u_0(x) = \begin{cases} 0, \ x < 0 \\ 1, \ x > 0 \end{cases} \end{cases}$$

Along with that consider the function

$$u(x,t) = \begin{cases} 0, \ x < \frac{1}{2}t \\ 1, \ x > \frac{1}{2}t \end{cases}$$

a. Show that this u is a weak solution to the Burgers' Equation initial value problem.

b. Show that this solution violates the entropy condition

$$f'(u^-) > s = \frac{d\hat{x}}{dt} > f'(u^+)$$

somewhere along the shock curve. Explain heuristically in terms of shock wave diagrams why this is an undesirable trait of the solution.