21632 Midterm Practice Problems

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1 Disclaimer

These problems do not necessarily reasonably represent the topics that could be on the actual exam. I pulled ideas from the lecture notes and textbook in design of these problems.

2 Differentiating Under the Integral Sign with Unit Normals

a. Let $U \subset \mathbb{R}^n$ be an open, bounded set and let $u \in C^1(\overline{U})$. Prove that

$$\int_U nu + Du \cdot x dx = \int_{\partial U} ux \cdot \nu d\mathbb{S}.$$

b. Now let w be a smooth function in $\mathbb{R}^n \times \mathbb{R}$ and show that for any $t > 0, x_0 \in \mathbb{R}^n$,

$$\frac{d}{dt}\int_{B(x_0,t)} \text{div}(u)(x \cdot \nu) d\mathbb{S} = \int_{B(x_0,t)} \text{div}(u)\left(n + \frac{\partial^2 x}{\partial \nu \partial t}\right) + (x \cdot \nu)\text{div}(u_t) + \sum_{i=1}^{n} u_{x_i x_i} x_i dx$$

where div denotes divergence with respect to spatial dimensions only, and ν is a unit normal vector.

Hint: Use part a with $u = \text{div}(w)$ and $U = B(x_0, t)$. You will want to break the divergence integrals into sums of n integrals and use the Leibniz Differentiation Rule somewhere. The point of this problem is to practice unifying many of the multivariate calculus identities we've put to use in class.

3 Abusing Linearity on Trigonometric-Exponential Models

Consider the boundary-value problem

$$\begin{cases} u_t + u_{tt} - 2u_{xx} = 0, \ (x,t) \in \mathbb{R} \times (0,\infty) \\ u(x,0) = \sin(x) + \frac{x}{2}, \ (x,t) \in (0,\infty) \times \{t=0\} \\ u(x,0) = \sin(x), \ (x,t) \in (-\infty,0] \times \{t=0\} \\ u_t(x,0) = \cos(x), \ (x,t) \in \mathbb{R} \times \{t=0\} \end{cases}$$

Show that there exists a $C^2(\mathbb{R} \times (0,\infty))$ solution to this boundary value problem of the form

$$u(x,t) = A\sin(x+t) + B\sqrt{t}e^{-\frac{x^2}{4t}} + C\frac{x}{\sqrt{t}}\int_0^\infty e^{-\frac{(x-y)^2}{4t}}dy$$

on $\mathbb{R} \times (0,\infty)$, where A, B, C are some real constants.

Hint: This PDE is the sum of two linear PDEs we have studied in the course. Decompose the PDE and boundary conditions into two separate, linear BVPs with which you are more familiar.

4 Uniqueness of Perturbations of the Heat Equation

For any $\epsilon > 0$, and a fixed $X > 0, T > 0$ consider the following boundary value problem.

$$\begin{cases} (u_\epsilon)_t - (u_\epsilon)_{xx} + \epsilon u_\epsilon = 0, \ (x,t) \in (0,X) \times (0,T) \\ u_\epsilon = 0, \ (x,t) \in (0,X) \times \{t=0\} \\ u_\epsilon = 0, \ (x,t) \in (0,X) \times \{t=T\} \end{cases}$$

We propose the following method to show uniqueness of Choose some nonnegative energy function $E_\epsilon(t)$ (in terms of ϵ) and show that there exists at most one solution $u_\epsilon \in C^2([0,X] \times [0,T])$ to the above boundary problem for any $\epsilon > 0$. Then use a limiting argument to show the following boundary value problem has at most one solution $u \in C^2([0,X] \times [0,T])$:

$$\begin{cases} u_t - u_{xx} = 0, \ (x,t) \in [0,X] \times (0,T] \\ u = 0, \ (x,t) \in (0,X) \times \{t=0\} \\ u = 0, \ (x,t) \in (0,X) \times \{t=T\} \end{cases}$$