Grad School Journeys: Nonlocal Edition

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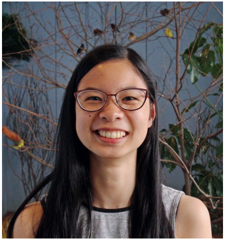
THE UNIVERSITY OF TENNESSEE KNOXVILLE
DEPARTMENT OF MATHEMATICS

Bio:

- Graduated from CMU with BS, MS in Mathematics in 2019
- Former Math Club and CMIMC officer
- Former TA for Concepts of Math, Matrix Theory, Math Studies Analysis II
- PhD candidate in combinatorics at University of California, San Diego



The Barrow Neurological Foundation funds research grants and raises awareness for brain aneurysms and other neurological conditions:
https://supportbarrow.rallybound.org/tributes/emilyzhu

The Trevor Project provides crisis support for at-risk LGBTQIA+ youth:
https://give.thetrevorproject.org/emilyzhu

The San Diego Zoo Wildlife Alliance funds research and education efforts worldwide for conservation efforts:
https://give.classy.org/emilyzhu_sdzwa

Brief Bio

- Born in Washington, D.C. but grew up in Pittsburgh
- Competed in math competitions in middle/high school
- Attended Carnegie Mellon from Fall 2015 through Fall 2018
- B.S. Mathematics, minor in Philosophy
- Hobbies include: blogging, hiking, board/card games

Undergrad Adventures

- Joined Steven Miller's research team in number theory and combinatorics in 2016 (still working with him today)
- Traveled to San Diego and Columbus to present work
- Worked at Expii part-time for Po-Shen Loh in 2019/2020

Go Vols!

- Joined University of Tennessee's math department in August 2019
- Expected graduation Spring 2024
- Concentration: numerical partial differential equations and optimal control

Whether to go to grad school?

To help you decide...

1. Get undergrad research or TA experience as a trial run
2. Does your "dream job" require a higher degree?
3. Do you enjoy being in a university environment?
4. Do you want to concentrate on studying one problem (or family of related problems) for an extended period of time?
5. Do you want to create new knowledge rather than just reusing or applying old knowledge?

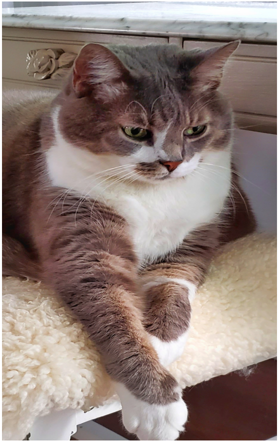
Grad school is an investment of your time to unlock opportunities for your professional future

Advice on picking grad schools

1. Focus on a program based on potential advisors, not the ranking!
2. Don't be surprised if your research interests change
3. Build mentor-mentee relationships now (good for letters of recommendation and a richer academic experience at CMU)
4. Let location constrain you as little as possible, especially if coming right out of undergrad
5. Aim for 8-10 programs of varying levels of competitiveness

Other advice for grad students

1. Don't stay up until 3 AM anymore, not worth burning yourself out
2. Think both in the short term and the long term for your research progress and other tasks
3. Network early and often, build a LinkedIn/ResearchGate profile, travel to conferences when possible

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Acknowledgments

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- Thanks to Jason Howell and the CMU Math Club officers for coordinating the talk
- Thanks to NSF grant 2111228 for financial support
- Thanks to the CMU Math Department and Steven J. Miller for kickstarting my growth as a research mathematician
- Thanks to Alisa Chang for helping me compile information on Emily Zhu

For our problem the nonlocal operator is

$$\mathcal{L}_\delta u(x) \;=\; \frac{1}{2}\int_{\Omega_\delta} \mathfrak{A}(x,y) k_\delta(|x-y|)\frac{Du(x,y)}{|x-y|}\frac{y-x}{|x-y|}dy$$

Nonlocal equations [or systems] take the form

$$\begin{cases} \mathcal{L}_\delta u \;=\; g, x \in \Omega \\ u \;=\; 0, x \in \Omega_\delta \setminus \Omega \end{cases}$$

Common in solid state mechanics, including peridynamics

Definition (PD)

Peridynamics (PD) is a nonlocal model for elasticity of solids that uses integrals over derivatives, attributed to Stewart A. Silling

Features:

- Exchanges derivatives in continuum models for integrals (helps address crack formation)
- Treats particles as having a bond between them (**bond-based model**)
- Range of interaction parameterized by δ, called **horizon**

Problem Statement

Find $(\overline{u_\delta}, \overline{g_\delta}) \in X_0 \times Z_{\text{ad}}$ such that

$$I(\overline{u_\delta}, \overline{g_\delta}) \ = \ \min_{g_\delta \in L^2, u_\delta \in X_0} \left\{ \int_\Omega F(x, u_\delta(x))dx + \frac{\lambda}{2} \int_\Omega \Gamma(x)|g(x)|^2 dx \right\},$$

over pairs $(u_\delta, g_\delta) \in X_0 \times Z_{\text{ad}}$ that satisfy some state equation (exact form TBD)

$$B_\delta(u_\delta, w_\delta) \ = \ \int_\Omega g_\delta(x) \cdot w_\delta(x), \ \forall w_\delta \in X_0.$$

where $\delta \geq 0$ is the degree of non-locality. Here $\overline{g_\delta}$ is an external force and $\overline{u_\delta}$ represents the displacement

Sample Candidate integrand

Example:

$$F(x, u(x)) = |u(x) - u_{\text{des}}(x)|^2$$

where u_{des} is the optimal shape of the material in space to fit a pre-determined hole as closely as possible

Goals

- Show existence and uniqueness of minimizers
- Consider behavior as $\delta \to 0^+$
- Discretize via FEMs
- Study simultaneous limit as $\delta, h \to 0^+$ (asymptotic compatibility)

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- Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $\Omega_\delta := \Omega \cup \{x, \mathrm{dist}(x, \partial\Omega) < \delta\}$
- $\Omega_\delta \setminus \Omega$ is non-local boundary
- $\mathcal{D}_\delta := (\Omega \times \Omega_\delta) \cup (\Omega_\delta \times \Omega)$
- Projected difference: $Du(x, y) := \frac{(u(x)-u(y))\cdot(x-y)}{|x-y|}$, nonlocal linearized strain (for vector-valued functions)
- Our material coefficient function is

$$\mathfrak{A}(x, y) \; := \; \frac{\mathfrak{a}(x) + \mathfrak{a}(y)}{2},$$

where there exist $a_{\min}, a_{\max} > 0$ so $a_{\min} \leq \mathfrak{a} \leq a_{\max}$ on Ω_δ.

Kernel sequence $\{k_\delta\}_{\delta>0} \subset L^1(\mathbb{R}^n)$ radial, integrable, non-negative, supported in $B(0,\delta)$ with

$$\int_{\mathbb{R}^n} k_\delta(\xi) d\xi \; = \; 1$$

Example: Truncated fractional kernels of form $k_\delta(\xi) \sim \frac{1}{|\xi|^{n+2s}}$

Nonlocal bi-linear form:

$$B_\delta(u,v) \;:=\; \frac{1}{2}\iint_{\mathcal{D}_\delta} \mathfrak{A}(x,y)k_\delta(x-y)\frac{Du(x,y)}{|x-y|}\frac{Dv(x,y)}{|x-y|}dxdy$$

Local bi-linear form:

$$B_0(u,v) \;:=\; C(n)\int_\Omega \mathfrak{a}(x)(2\langle \mathrm{Sym}(\bigtriangledown u), \mathrm{Sym}(\bigtriangledown v)\rangle_F + \mathrm{div}(u)\mathrm{div}(v))dx,$$

with $C(n) = \frac{1}{(n+2)(n+4)}$

Inner products denoted $\langle \cdot,\cdot\rangle_Y$; L^2-inner product denoted $\langle \cdot,\cdot\rangle$

Our function space is based on B_δ :

$$X(\Omega_\delta; \mathbb{R}^n) \ := \ \{u|_\Omega \in L^2(\Omega; \mathbb{R}^n), B_\delta(u,u) < \infty\}$$

Version with zero non-local boundary data:

$$X_0(\Omega_\delta; \mathbb{R}^n) \ := \ \{u \in X(\Omega_\delta; \mathbb{R}^n), u = 0 \text{ in } \Omega_\delta \setminus \Omega\}$$

These are Hilbert spaces!

Cost Functional Assumptions

$$I(u,g) \ := \ \frac{1}{2}\|u-u_{\text{des}}\|^2_{L^2(\Omega;\mathbb{R}^n)}+\frac{\lambda}{2}\|g\|^2_{L^2(\Omega;\mathbb{R}^n)}$$

Here Z_{ad} is a nonempty, closed, convex, and bounded subset of $L^2(\Omega;\mathbb{R}^n)$, taking the form

$$Z_{\text{ad}} \ := \ \{z \in L^2(\Omega;\mathbb{R}^n), a \preceq z \preceq b\}$$

Here $[a]_i \le [b]_i$ for all $i \in \{1,2,\ldots,n\}$ with $a = ([a]_1,\ldots,[a]_n)$ and $b = ([b]_1,\ldots,[b]_n)$ being vector fields in $L^2(\Omega;\mathbb{R}^n)$, $\lambda \ge 0$.

State equation is well-posed!

Theorem (Existence and Uniqueness for State Equation)

For any $g_\delta \in L^2$, there exists a unique $u_\delta \in X_0$ such that the state system

$$B_\delta(u_\delta, w_\delta) \;=\; \langle g_\delta, w_\delta \rangle$$

is satisfied for all $w_\delta \in X_0$. Furthermore, we have the stability estimate

$$\|u_\delta\|_{X(\Omega_\delta;\mathbb{R}^n)} \lesssim \; \|g_\delta\|_{X(\Omega_\delta;\mathbb{R}^n)^*}$$

for some constant independent of δ.

Minimization Problem

Goal: find $(\overline{u_\delta}, \overline{g_\delta}) \in X_0 \times L^2$ minimizing

$$I(u_\delta, g_\delta) = \frac{1}{2}\|u_\delta - u_{\text{des}}\|^2_{L^2(\Omega;\mathbb{R}^n)} + \frac{\lambda}{2}\|g_\delta\|^2_{L^2(\Omega;\mathbb{R}^n)}$$

subject to: $\lambda \geq 0$, $g_\delta \in Z_{ad} \subset L^2$ and $(u_\delta, g_\delta) \in X_0 \times L^2$ solving

$$B_\delta(u_\delta, v_\delta) = \int_\Omega g_\delta(x) \cdot v_\delta(x) dx \quad \forall v_\delta \in X_0$$

Well-posedness of optimal control problem

Theorem (Well-posedness)

There exists $(\overline{u_\delta}, \overline{g_\delta}) \in X_0(\Omega_\delta; \mathbb{R}^n) \times Z_{ad}$ *minimizing*

$$I(u_\delta, g_\delta) \;=\; \frac{1}{2}\|u_\delta - u_{des}\|^2_{L^2(\Omega;\mathbb{R}^n)} + \frac{\lambda}{2}\|g_\delta\|^2_{L^2(\Omega;\mathbb{R}^n)},$$

where $\overline{u_\delta} \in X_0$ *solves*

$$B_\delta(u_\delta, v_\delta) = \int_\Omega g_\delta(x) \cdot v_\delta(x) dx \quad \forall v_\delta \in X_0$$

Furthermore, if F *is strictly convex or* $\lambda > 0$*, then the minimizer is unique.*

Use compactness to apply direct method

Find $(\overline{u_{\delta,h}}, \overline{g_{\delta,h}}) \in X_{\delta,h} \times Z_h$ such that

$$I(\overline{u_{\delta,h}}, \overline{g_{\delta,h}}) = \min_{u_{\delta,h} \in X_{\delta,h},\ g_{\delta,h} \in Z_h} I(u_{\delta,h},\ g_{\delta,h}),$$

over pairs $(u_{\delta,h}, g_{\delta,h}) \in X_{\delta,h} \times Z_h$ that satisfy

$$B_\delta(u_{\delta,h}, v_{\delta,h}) \ = \ \langle g_{\delta,h}, v_{\delta,h} \rangle, \qquad \forall v_{\delta,h} \in X_{\delta,h}.$$

Recap:

$$I(u_{\delta,h}, g_{\delta,h}) \ := \frac{1}{2} \| u_{\delta,h} - u_{\text{des}} \|^2_{L^2(\Omega;\mathbb{R}^n)} + \frac{\lambda}{2} \| g_{\delta,h} \|^2_{L^2(\Omega;\mathbb{R}^n)}$$

Notation

- Mesh family: $\{\mathscr{T}_h\}_{h>0}$ (discretizing Ω_δ) shape-regular and quasi-uniform
- Piecewise polynomials of degree m (with respect to our mesh):

$$\mathcal{P}_m(T;\mathbb{R}^n) := \left\{ \sum_{\alpha\in\mathbb{N}_0^n \,:\, \sum_{i=1}^n \alpha_i \le m} v_\alpha x_1^{\alpha_1}\cdots x_n^{\alpha_n} \;\middle|\; v_\alpha \in \mathbb{R}^n,\ (x_i)_{i=1}^n \in T \right\}$$

- Discretized state space:
  $X_{\delta,h} := \{w_h \in C^0(\overline{\Omega_\delta};\mathbb{R}^n) \mid w_h|_T \in \mathcal{P}_1(T;\mathbb{R}^n)\ \forall T \in \mathscr{T}_h, w_h = 0 \text{ on } \Omega_\delta \setminus \Omega\}$
- Discretized control space: $Z_h := \{z_h|_T \in \mathcal{P}_0(T;\mathbb{R}^n)\ \forall T \in \mathscr{T}_h\}$

Theorem (Convergence of Controls)

Assume that $\overline{g_\delta}$ is the optimal control associated with the nonlocal continuous problem, and $\overline{g_{\delta,h}}$ be the discrete optimal control. Then we have the convergence

$$\|\overline{g_\delta} - \overline{g_{\delta,h}}\|^2_{L^2(\Omega;\mathbb{R}^n)} \lesssim \omega(h) + \left(\inf_{v_{\delta,h} \in X_{\delta,h}} [\overline{u_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)} \right)^2 + \left(\inf_{v_{\delta,h} \in X_{\delta,h}} [\overline{p_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)} \right)^2 .$$

Theorem (Full Norm Solution Convergence)

In the setting of our problem formulation,

$$\|\overline{u_\delta} - \overline{u_{\delta,h}}\|_{X(\Omega_\delta;\mathbb{R}^n)} \lesssim \omega(h) + \inf_{v_{\delta,h}\in X_{\delta,h}} \|\widehat{u_\delta} - v_{\delta,h}\|_{X(\Omega_\delta;\mathbb{R}^n)} + \inf_{v_{\delta,h}\in X_{\delta,h}} [\overline{u_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)} + \inf_{v_{\delta,h}\in X_{\delta,h}} [\overline{p_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)};$$

$$\|\overline{p_\delta} - \overline{p_{\delta,h}}\|_{X(\Omega_\delta;\mathbb{R}^n)} \lesssim \omega(h) + \inf_{v_{\delta,h}\in X_{\delta,h}} \|\widehat{p_h} - v_{\delta,h}\|_{X(\Omega_\delta;\mathbb{R}^n)} + \inf_{v_{\delta,h}\in X_{\delta,h}} \|\widehat{u_\delta} - v_{\delta,h}\|_{X(\Omega_\delta;\mathbb{R}^n)} + \inf_{v_{\delta,h}\in X_{\delta,h}} [\overline{u_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)} + \inf_{v_{\delta,h}\in X_{\delta,h}} [\overline{p_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)}.$$

NOTE: $\omega(h)$ is a stability term from finite element approximations, i.e. $\lim_{h\to 0^+}\omega(h) = 0$.

Links

- Link to my blog: https://medium.com/@joshuasiktar
- Link to paper preprint: https://arxiv.org/pdf/2304.09328.pdf
- Link to LinkedIn: https://www.linkedin.com/in/joshuasiktar1/
- Link to Emily Zhu's webpage and source of photograph: https://mathweb.ucsd.edu/~e9zhu/