Recurrence relations within n-gap sequences for integer decompositions

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CANT, May 2019

Previous Work

Definition

The Fibonacci numbers are the sequence define by $F_k = F_{k-1} + F_{k-2}$ with initial conditions $F_1 = 1, F_2 = 1$.

Theorem (Zeckendorf, 1972)

Every positive integer n can be expressed uniquely as a sum $n = \sum_k a_k F_k$, where $a_k \in \{0, 1\}$ and if $a_i = a_j = 1$ for some $i < j$, then there is a k with $i < j < k$ such that $a_k = 0$.

Theorem (Even Fibonacci Decompositions)

Every positive integer n can be expressed uniquely as a sum $n = \sum_k a_k F_{2k}$, where $a_k \in \{0, 1, 2\}$ and if $a_i = a_j = 2$ for some $i < j$, then there is a k with $i < j < k$ such that $a_k = 0$.

Motivation

What happens when we look at integer decompositions for other sequences?

We look for an inherent relationship between the sequences and the conditions for *unique* decompositions, especially coefficients for multiplication and excluded terms.

PLRS

Definition (Miller, Wang, 2012)

A Positive Linear Recurrence Sequence (PLRS) is a sequence of integers $\{H_n\}_{n=1}^{\infty}$ with the following properties:

1. There exist nonnegative integers $L, c_1, ..., c_L$ such that:

$$H_n = c_1 H_{n-1} + c_2 H_{n-2} + ... + c_L H_{n-L}$$

   where $L, c_1, c_L > 0$.

2. For $n \in [2, L]$, we have:

$$H_n = c_1 H_{n-1} + c_2 H_{n-2} + ... + c_{n-1} H_1 + 1$$

Theorem (Miller, Wang, 2012)

Let $\{H_n\}_{n=1}^{\infty}$ be a PLRS. Then for each positive integer $N > 0$ there exists a unique legal decomposition dependent on N.

n-gap Sequences

Question: What happens when we look at every n$^{\text{th}}$ term in the sequence? Can we still recover *unique* decompositions?

We will call these the n-gap (sub)sequences, where n is the distance between consecutive terms of the new sequence within the old sequence.

Definition

Let $\{G_k\}_{k=1}^{\infty}$ be an integer sequence and $n \in \mathbb{Z}^+$. The n-gap subsequence of $\{G_k\}$ is $\{G_{nk}\}_{k=1}^{\infty}$.

Moreover, the first term of the subsequence can be any of the first n terms. We can express this via $\{G_{nk+m}\}_{k=1}^{\infty}$, where $0 \leq m < n$.

n-gap Fibonaccis

Example

The following are the first five n-gap sequences for the Fibonacci numbers:

- **0-gap:** 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89...
- **1-gap:** 1, 2, 5, 13, 34, 89, 233, 610, 1597...
- **2-gap:** 1, 3, 13, 55, 233, 987, 4181...
- **3-gap:** 1, 5, 34, 233, 1597...
- **4-gap:** 1, 8, 89, 987...
- **5-gap:** 1, 13, 233, 4181...

Note that a 0-gap sequence is exactly the original sequence.

A Closer Look at the 2-gap Fibonacci Sequence

1, 5, 21, 89, 377, 1597...

To decompose positive integers (ignoring uniqueness), we need a coefficient of at least 4.

In fact, the numbers in this sequence are related via the following recurrence:

$$a_n = 4 \cdot a_{n-1} + a_{n-2}$$

Generalizing

Sequence:	Corresponding recurrence:
1-gap: 1, 2, 5, 13, 34, 89...	$a_n = 3a_{n-1} - a_{n-2}$
2-gap: 1, 3, 13, 55, 233...	$a_n = 4a_{n-1} + a_{n-2}$
3-gap: 1, 5, 34, 233, 1597...	$a_n = 7a_{n-1} - a_{n-2}$
4-gap: 1, 8, 89, 987...	$a_n = 11a_{n-1} + a_{n-2}$
5-gap: 1, 13, 233, 4181...	$a_n = 18a_{n-1} - a_{n-2}$

Note that 3, 4, 7, 11, 18... are the Lucas numbers.

n-gap Recurrences

Lemma

Let G_n be a recurrence relation defined via $G_n = G_{n-1} + G_{n-2}$ with initial conditions $G_1 = a, G_2 = b$, where $a, b \in \mathbb{Z}^+$. Then,

$$G_{nk+m} = a_n \cdot G_{n(k-1)+m} + (-1)^{n+1} \cdot G_{n(k-2)+m}$$

where n is the length of the gap and $a_n = \phi^n$ rounded to the nearest integer (i.e. the n$^{\text{th}}$ Lucas number).

Our Result

Theorem

For odd n, the n-gap subsequence recurrences are the PLRSs necessary for unique decompositions of positive integers.

The condition for odd n comes from the $(-1)^{n+1}$ coefficient. By definition, a PLRS must have all nonnegative coefficients, so an odd n prevents a negative term.

Conjecture

We can naturally extend this to more general sequences.

Conjecture

Let H_n be a recurrence relation defined via $H_n = \alpha \cdot H_{n-1} + \beta \cdot H_{n-2}$ with initial conditions $H_1 = a, H_2 = b$, where $\alpha, \beta, a, b \in \mathbb{Z}^+$. Then,

$$H_{nk+m} = c_{n+2} \cdot H_{n(k-1)+m} - (-\beta)^{n+1} \cdot H_{n(k-2)+m}$$

where n is the length of the gap and c_{n+2} comes from the sequence $c_n = \alpha \cdot c_{n-1} + \beta \cdot c_{n-2}$ with $c_1 = 2, c_2 = \alpha$.

Thank you!