DERIVATIVES AND ESTIMATES: THEOREMS IN COMPLEX ANALYSIS

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1. Introduction

This document is designed as a reference for theorems in complex analysis (University of Tennessee-Knoxville, MATH 546 Spring 2020) on estimating derivatives. Most of the theorems are useful for solving prelim-style problems studying holomorphic functions on a disc, as well as problems pertaining to harmonic function theory for real or complex variables. The ordering closely follows the exposition of [Sa].

Throughout G refers to an open connected subset of $\mathbb{C}$ unless stated otherwise.

2. Estimates

Theorem 2.1 (Maximum Modulus Principle)**.** *(Sarason p. 90) Let f be a nonconstant holomorphic function in G mapping onto $\mathbb{C}$. Then $|f|$ does not attain a local maximum in G.*

Theorem 2.2 (Schwarz Lemma)**.** *(Sarason p. 91-92) Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic with $f(0) = 0$. Then $|f(z)| \leq |z|$ in all of $\mathbb{D}$, and the inequality is strict everywhere except the origin, unless $f(z) = \lambda z$ for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Furthermore, $|f'(0)| \leq 1$.*

Theorem 2.3 (Pick's Lemma)**.** *(Sarason p. 92) Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic. If $z, w \in \mathbb{D}$ then*

$$\left|\frac{f(z) - f(w)}{1 - f(z)\overline{f(w)}}\right| \leq \left|\frac{z - w}{1 - z\overline{w}}\right|$$

and the inequality is strict for $z \neq w$ except when f is a linear fractional transformation mapping $\mathbb{D}$ onto $\mathbb{D}$.

3. Harmonic Functions

Theorem 3.1. *(Sarason p. 16) The function f is differentiable in the complex sense at z_0 iff u and v are differentiable in the real sense at z_0 and*

$$\frac{\partial u}{\partial x}(z_0) = \frac{\partial v}{\partial y}(z_0)$$
$$\frac{\partial u}{\partial y}(z_0) = -\frac{\partial v}{\partial x}(z_0)$$

If all this is true, it also holds that

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$$\frac{\partial f}{\partial z}(z_0) = \frac{\partial u}{\partial x}(z_0) + i\frac{\partial v}{\partial x}(z_0) = \frac{\partial u}{\partial y}(z_0) - i\frac{\partial v}{\partial y}(z_0)$$

Theorem 3.2. *(Sarason p. 16, 24) If $f : G \to \mathbb{C}$ is holomorphic then it is also harmonic. Furthermore, its real and complex parts satisfy the Cauchy-Riemann Equations and are themselves harmonic.*

Theorem 3.3 (Liouville Theorem)**.** *(Sarason p. 86) An entire bounded function is constant.*

Theorem 3.4 (Identity Theorem Complex-Valued)**.** *(Sarason p. 89) Let $f, g : G \to \mathbb{C}$ be holomorphic. If $f(z) = g(z)$ in a subset of G containing a limit point of G, then $f(z) = g(z)$ for all $z \in G$.*

Theorem 3.5 (Existence of Harmonic Conjugates)**.** *(Sarason p. 93) Let $u : G \to \mathbb{R}$ where G is an open convex subset of $\mathbb{C}$. Then there is a holomorphic function g, unique up to addition of an imaginary constant, where $u = \Re g$.*

Theorem 3.6 (Regularity)**.** *(Sarason p. 94) Any harmonic function is smooth.*

Theorem 3.7 (Mean-Value Property)**.** *(Sarason p. 94) Let u be a harmonic function in G an open subset of $\mathbb{C}$, let $z_0 \in G$, and $0 < r < dist(z_0, \mathbb{C} \setminus G)$. Then*

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{it})dt$$

Theorem 3.8 (Identity Theorem for Harmonic Functions)**.** *(Sarason p. 94) If G is a connected open subset of $\mathbb{C}$ and if u and v are harmonic functions in G that are equal on a nonempty open subset of G, then $u = v$ throughout G.*

Theorem 3.9 (Maximum Principle for Harmonic Functions)**.** *(Sarason p. 95) Let u be a nonconstant real-valued harmonic function in the connected open subset $G \subset \mathbb{C}$. Then u does not attain a local maximum in G.*

REFERENCES

[Sa] D. Sarason, “Complex Function Theory," Second Edition. American Mathematical Society, 2000.