# Asymptotic Analysis For Lattice Walks Derived From Zeckendorf Decompositions

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## Overview

- Introduction to the Lattice Walks
- Overview of Main Results and Simulations
- Technical Lemmas
- Proof of Main Results
- Future Work

### Definition (Fibonacci Numbers)

The **Fibonacci Numbers** are a sequence defined recursively with $F_{n+1} = F_n + F_{n-1}\ \forall n \geq 2$ where $F_1 = 1$ and $F_2 = 2$.

**Beginning of sequence:** $1, 2, 3, 5, 8, 13, 21, 34, 55, \dots$

### Definition (Zeckendorf Decompositions)

A **Zeckendorf Decomposition** is a way to write a natural number as the sum of non-adjacent Fibonacci Numbers. They also give an alternative definition for the Fibonacci Numbers.

### Theorem (Zeckendorf's Theorem)

*Every natural number has a unique Zeckendorf Decomposition.*

Example (Greedy Algorithm):

- 335
- $335 = 233 + 102$
- $335 = 233 + 89 + 13$

## Definition (Simple Jump Paths (in 2D))

A **simple jump path** is a path on the lattice grid where each movement on the lattice grid consists of at least one unit movement to the left and one unit movement downward.

**Examples of simple jump paths** (from $(7, 7)$ to $(0, 0)$)

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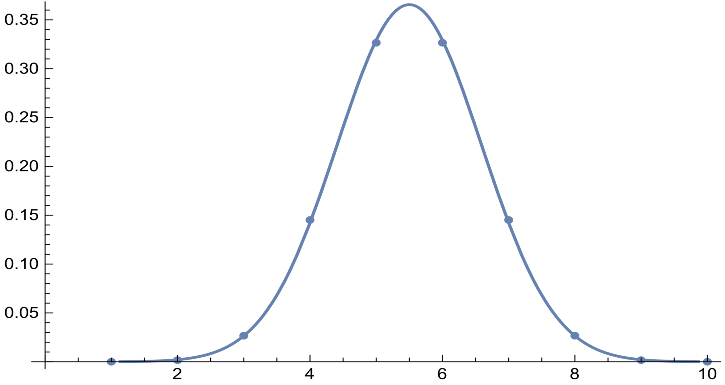
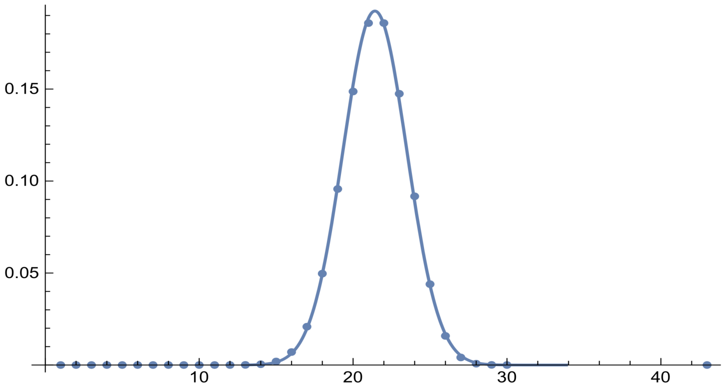
- We count simple jump paths from $(a, b)$ to $(0, 0)$, where $a, b \in \mathbb{N}^+$
- Let the number of simple jump paths from $(a, b)$ to $(0, 0)$ be denoted $s_{a,b}$; always include $(a, b)$ and $(0, 0)$
- Let the number of simple jump paths from $(a, b)$ to $(0, 0)$ with $k$ steps be denoted $t_{a,b,k}$
- Analogue in $d = 1$ resembles base-2 expansion

**Theorem (E. Chen, R. Chen, L. Guo, C. Jiang, S.M., J.S., P.Y.)**

***Simple Path Gaussianity on $d$-dimensional Lattice:*** *Let $n$ be a positive integer, and consider the distribution of the number of summands among all simple jump paths with starting point $(p_1, p_2, \dots, p_d)$ where $1 \leq p_1, p_2, \dots, p_d \leq n$, and each path represents a (not necessarily unique) decomposition of some positive number. This distribution converges to a Gaussian as $n \to \infty$ with mean $\frac{1}{2}n + 1$ and standard deviation $\frac{\sqrt{n}}{2\sqrt{d}}$.*




- Easiest to visualize what is going on when $d = 2$
- Simple jump paths over a square lattice for $n = 10$, starting point $(10, 10)$
- Plotted points represent $\{t_{10,10,k}\}_{k=1}^{10}$, with best-fit Gaussian




- Simple jump paths over a rectangular lattice with starting point $(70, 30)$
- Plotted points represent $\{t_{30,70,k}\}_{k=1}^{30}$, with best-fit Gaussian

**Lemma (Simple Jump Path Partition Lemma)**

*If $s_d(n)$ denotes the number of $d$-dimensional paths from $(n, n, \ldots, n)$ to the origin and $t_d(n, k)$ denotes the number of such paths with $k$ steps, then $s_d(n) = \sum_{k=1}^{n} t_d(n, k)$.*

- Here $t_d(n, k)$ denotes the number of simple jump paths of $k$ steps starting from point $(n, n, ..., n)$ in $d$-dimensions

**Lemma (The Cookie Problem)**

*The number of ways of dividing C identical cookies among P distinct people is* $\binom{C+P-1}{P-1}$.

**Lemma (Enumerating Simple Jump Paths in d-dimensions)**

$$\forall n \in \mathbb{N},\ 1 \leq k \leq n,\ t_d(n,k) \ = \ \binom{n-1}{k-1}^d.$$

- Every $\binom{n-1}{k-1}$ is the number of ways to group $k$ objects into $n$ nonempty groups
- Groupings are independently determined, use Cookie Problem lemma

Useful formulas and notation:

- $p(x_k)$: probability of event $x_k$ occurring, one of finitely many values (events)
- Density function: $f_d(k, n) := \frac{t_d(n+1,k+1)}{s_d(n+1)} = \frac{\binom{n}{k}^d}{s_d(n+1)}$
- Mean (discrete): $\mu = \sum x_k p(x_k)$
- Variance (discrete): $\sigma^2 = \sum (x_n - \mu)^2 p(x_n)$
- Gaussian (continuous): Density $(2\pi\sigma^2)^{-1/2} \exp(-(x-\mu)^2/2\sigma^2)$

**Lemma (Mean on $d$-dimensional Lattice)**

$\forall n \in \mathbb{N}^+, \mu_d(n+1) = \frac{1}{2}n + 1 \sim \frac{n}{2}.$

- The mean is independent of $d$

**Lemma (Standard Deviation on Square Lattice)**

$\forall n \in \mathbb{N}^+, \sigma_1(n+1) = \frac{\sqrt{n}}{2}, \sigma_2(n+1) = \frac{n}{2\sqrt{2(n-1)}} \sim \frac{\sqrt{n}}{2\sqrt{2}}.$

- Calculate using definition of first moment (mean) and second moment (standard deviation)
- Use index shift: $\sum_{k=1}^{n+1}$ becomes $\sum_{k=0}^{n}$, $\binom{n}{k-1}$ becomes $\binom{n}{k}$
- Use binomial expansion and standard techniques for evaluating binomial coefficients

## Lemma (Standard Deviation on d-dimensional Lattice)

$\forall d \geq 2, n \in \mathbb{N}^+, \sigma_d(n+1) \leq \sigma_1(n+1) \leq \frac{\sqrt{n}}{2}$

- We weren't able to find closed-form expression for $\sigma$ in higher dimensions
- For example, the evaluation of $\sum_{k=0}^{n} k^d \binom{n}{k}^d$ cannot be generalized for $d > 2$
- The variance decreases as $d$ increases, and it is largest when $d = 1$, proven using symmetry of binomial coefficients
- In fact, it holds that $\sigma_d(n+1) \sim \frac{\sqrt{n}}{2\sqrt{d}}$

**Lemma (Bounding the random variable)**

*Consider all simple jump paths from $(n+1, n+1, ..., n+1)$ to the origin in $d$-dimensions. If $K$ is the random variable denoting the number of steps in each path, then the probability that $K$ is at least $\frac{n^{\epsilon}\sqrt{n}}{2}$ from the mean is at most $n^{-2\epsilon}$.*

- By Chebyshev's Inequality, $\text{Prob}(|K - \mu_d| \geq n^{\epsilon}\sigma_d(n+1)) \leq \frac{1}{n^{2\epsilon}}$
- As $\sigma_d \leq \frac{\sqrt{n}}{2}$ by the previous lemma, we only decrease the probability on the left if we replace $\sigma_d(n+1)$ with $\frac{\sqrt{n}}{2}$
- If we write $K$ as $\mu_d(n+1) + l \cdot \frac{\sqrt{n}}{2}$, then with probability tending to 1 we may assume $|l| \leq n^{\epsilon}$

### Theorem (Simple Path Gaussianity on $d$-dimensional Lattice)

*Let $n$ be a positive integer, and consider the distribution of the number of summands among all simple jump paths with starting point $(p_1, p_2, \dots, p_d)$ where $1 \leq p_1, p_2, \dots, p_d \leq n$, and each distribution represents a (not necessarily unique) decomposition of some positive number. This distribution converges to a Gaussian as $n \to \infty$ with mean $\frac{1}{2}n + 1$ and standard deviation $\frac{\sqrt{n}}{2\sqrt{d}}$.*

- Write $k$ as $\mu_d(n+1) + l \cdot \frac{\sqrt{n}}{2}$, $l$ is the number of standard deviations from the mean
- Density function: $f_d(n+1, k+1) := \frac{t_d(n+1,k+1)}{s_d(n+1)} = \frac{\binom{n}{k}^d}{s_d(n+1)}$
- Use Stirling's Approximation on each factor: $m! \ \sim \ m^m e^{-m}\sqrt{2\pi m}$

- End result of Stirling expansion is $f_d(n+1,k+1) = \frac{2^{dn} n^{d/2}}{s_d(n+1)} \left( \frac{n^n}{2^n k^k (n-k)^{n-k} \sqrt{2\pi k(n-k)}} \right)^d \cdot \left(1 + O\left(\frac{1}{n}\right)\right)$
- Since $k$, $n-k$ are close to $n/2$, the main term becomes

$$f_{main} := \frac{n^n}{2^n k^k (n-k)^{n-k} \sqrt{2\pi k(n-k)}} = \frac{1}{\sqrt{\frac{\pi n^2}{2}}} \cdot \frac{1}{\left(1 - \frac{l}{\sqrt{n}}\right)^{\frac{n-l\sqrt{n}+1}{2}} \left(1 + \frac{l}{\sqrt{n}}\right)^{\frac{n+l\sqrt{n}+1}{2}}}$$

- Denote the denominator of the second fraction as $q_{n+1}$, approximate it using Taylor expansion

- Eventually we get $q_{n+1} = e^{\frac{(k-\mu_d(n+1))^2}{n/2}} \cdot e^{O(n^{-1/6})}$
- Then, for $|l| \leq n^{1/9}$,
  $f_d(n+1, k+1) = \frac{2^{dn} n^{d/2}}{s_d(n+1)(\pi n^2/2)^{d/2}} \cdot e^{-\frac{d(k-\mu_d(n+1))^2}{n/2}} e^{O(n^{-1/6})}$
- The second exponential is negligible as $n \to \infty$; the first exponential is Gaussian with mean $\mu_d(n+1)$ and variance $\sigma_d(n+1)^2 = \frac{n}{4d}$
- The normalization constant is $s_d(n+1) \sim 2^{dn}(\frac{\pi n}{2})^{\frac{1-d}{2}} d^{-\frac{1}{2}}$

## Definition (Generalized Jump Paths (in 2D))

A **generalized jump path** is a path on the lattice grid where each movement on the lattice grid consists of either at least one unit movement to the left or one unit movement downward.

**Examples of generalized jump paths** (from $(7,7)$ to $(0,0)$)

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**Theorem (E. Fang, J.J., Z. Lee, D. Li, E. Lu, S.M., D.S. J.S.)**

***Generalized Path Gaussianity on 2-dimensional Lattice:***
*Let $g((p,q),k)$ denote the number of generalized jump paths from the point $(p,q)$ using exactly $k$ moves. As $p,q \to \infty$, $g((p,q),k)$ is Gaussian with respect to $k$.*

- $g(\mathbf{p}, k)$ is Generalized Jump Paths from $\mathbf{p}$ with $k$ moves
- $u(\mathbf{p}, k)$ counts paths that don't necessarily end at (0,0).
- $u(\mathbf{p}, k) = g(\mathbf{p}, k) + g(\mathbf{p}, k+1)$

In 2 dimensions,

$$\begin{aligned} u((p,q),k) = {} & u((p,q-1),k) + u((p,q-1),k-1) \\ & + u((p-1,q),k) + u((p-1,q),k-1) \\ & - u((p-1,q-1),k) - u((p-1,q-1),k-1) \end{aligned}$$

- Let $F_{p,q}(x) = u((p,q),k)x^k$

**Claim**

$$F_{p,q}(x) = (1+x)^p \sum_{k=0}^{q} \binom{q}{k} \binom{p+k}{k} x^k$$

# Combinatorics Method

- Let $r(\mathbf{p}, n)$ be defined identically to $g(\mathbf{p}, n)$ but allowing stationary points
- Let $s(\mathbf{p}, n, k)$ correspond to $r(\mathbf{p}, n)$ where there are at least $k$ stationary points

By Stars and Bars,

$$r(\mathbf{p}, n) = \prod_{i=1}^{d} \binom{p_i + n - 1}{p_i}$$

- Observe $s(\mathbf{p}, n, k) = \binom{n}{k} r(p, n-k)$

Then by inclusion-exclusion,

$$g(\mathbf{p}, n) = \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} r(\mathbf{p}, n-k)$$

# Combinatorics Method

## Current Result

In 2-D,

$$g((p,q),n) = \sum_{i=0}^{n} (-1)^k \binom{n}{i} \binom{(p-1)+n-i}{(n-1)-i} \binom{(q-1)+n-i}{(n-1)-i}$$

where WLOG $p \leq q$

# Simplification

Inner term counts $(S, T, U)$ such that

- $S \subseteq [n], T \subseteq [p+n-1] \setminus S, U \subseteq [q+n-1] \setminus S$
- $|S| + |T| = |S| + |U| = n - 1$

Define $f$ where $f$ toggles minimum term of $S \cup (U \cap T)$

- $f$ is it's own involution
- $f$ flips parity of $|S|$
- Ordered pairs defined on $f$ sum to 0
- **Only need to sum if $f$ not well-defined**

# Combinatorics Method

Let the set where $f$ is not well-defined be $E$. Then, we can conclude

$$\sum_{i=0}^{n}(-1)^k\binom{n}{i}\binom{(p-1)+n-i}{(n-1)-i}\binom{(q-1)+n-i}{(n-1)-i}=|E|$$

$f$ is not well-defined if and only if

- $S=\varnothing$
- $T\cap U\cap[n]=\varnothing$

Basic combinatorial arguments then yield

$$g((p,q),n)=\sum_{i=0}^{n-1}\binom{p-1}{i}\binom{p-1+n-i}{p}\binom{q}{n-i-1}$$

# Combinatorics Method

- Use $u((p,q),n) = g((p,q),n) + g((p,q),n+1)$

$$u((p,q),n) = \sum_{i=0}^{n} \binom{p}{i} \binom{p+n-i}{p} \binom{q}{n-i}$$

- Plugging into $F_{p,q}(x)$:

$$F_{p,q}(x) = (1+x)^p \sum_{k=0}^{q} \binom{q}{k} \binom{p+k}{k} x^k$$

## Setup

- $X_{p,q}$ is random variable counting length of path
- $A, B$ random variables,
  $P(A = k) \propto \binom{p}{k}, P(B = k) \propto \binom{q}{k}\binom{p+k}{k}$
- $X_{p,q} = A_{p,q} + B_{p,q}$ by previous result

## Goals

**Well Known**

$A$ is Gaussian with mean $\frac{p}{2}$, standard dev $\frac{p}{4}$

**Theorem**

$B$ is Gaussian with mean $\frac{q-p+\sqrt{p^2+6pq+q^2}}{4}$

The proofs are routine calculations.

# Outline

- Use Stirling's Approximation
- Set $k = an + t\sqrt{n}$ where $a$ is mean and standard deviation is $O(\sqrt{n})$.
- Taylor Expansion about $\frac{t}{\sqrt{n}}$
- Show probability $|t| > n^{0.1} \to 0$ as $n \to \infty$

# Final Results

- The number of generalized jump paths is Gaussian with respect to the number of jumps.
- Mean: $\frac{p+q}{4} + \frac{\sqrt{p^2+6pq+q^2}}{4}$
- Variance: $\frac{p+q}{8} + \frac{(p+q)^2}{8\sqrt{p^2+6pq+q^2}}$

# Future Work

- Work out expected Gaussianity result for compound paths in higher dimensions
- Investigate rates of convergence to Gaussian
- What happens if we allow points on lattice to be visited more than once?

## Directly Related Papers


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- E. Fang, J. Jenkins, Z. Lee, D. Li, E. Lu, S.J. Miller, D. Salgado, J.M. Siktar *Central Limit Theorems for Compound Paths on the 2-Dimensional Lattice* (URL: `https://arxiv.org/pdf/1906.10645.pdf`)

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# Thank You


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