Asymptotic Compatibility of Parameterized Optimal Design Problems

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Problem statement

Find $(\overline{\mathfrak{a}_\delta}, \overline{u_\delta}) \in \mathcal{H} \times X_0(\Omega_\delta)$ such that

$$J(\overline{\mathfrak{a}_\delta}, \overline{u_\delta}) \;=\; \min_{\mathfrak{a}\in\mathcal{H}, u\in X_0} J(\mathfrak{a}, u),$$

The minimization is over pairs $(\mathfrak{a}, u) \in \mathcal{H} \times X_0(\Omega_\delta)$ that satisfy

$$\mathcal{L}_{\delta,\mathfrak{a}} u(x) \;=\; g(x) \text{ a.e. } x \in \Omega$$

- $\delta \geq 0$ is the degree of nonlocality;
- g is a fixed external force;
- $\overline{\mathfrak{a}_\delta}$ is a material coefficient (design);
- $\overline{u_\delta}$ represents the displacement (state);

Goals

- Show existence of minimizers
- Prove variational convergence to local problem, $h \geq 0$ fixed and $\delta \to 0^+$
- Discretize via FEM, $\delta \geq 0$ fixed and $h \to 0^+$
- Study simultaneous limit as $\delta, h \to 0^+$ (asymptotic compatibility)

Notation

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain.

- Design class:

$$\mathcal{H} \;:=\; \{\mathfrak{a} \in L^\infty(\Omega) \mid a_{\min} \;\leq\; \mathfrak{a}(x) \;\leq\; a_{\max} \text{ a.e. } x \in \Omega\}$$

  where $0 < a_{\min} < a_{\max}$
- k_δ is a non-negative kernel with suitable properties
- Nonlocal bi-linear form:

$$B_{\delta,\mathfrak{A}}(u,v) \;:=\; \iint_{\mathcal{D}_\delta} \mathfrak{A}(x,y)\frac{k_\delta(x-y)}{|x-y|^2}(u(x)-u(y))(v(x)-v(y))dxdy$$

  where $\mathcal{D}_\delta := (\Omega_\delta \times \Omega) \cup (\Omega \times \Omega_\delta)$ and $\mathfrak{A}(x,y) := \frac{\mathfrak{a}(x)+\mathfrak{a}(y)}{2}$
- Nonlocal function space:

$$X_0(\Omega_\delta) := \{u \in L^2(\Omega) \mid B_{\delta,\mathfrak{A}}(u,u) < \infty, u = 0 \text{ on } \Omega_\delta \setminus \Omega\}$$

  where $\Omega_\delta := \Omega + B(0,\delta)$

Outline

Existence of nonlocal optimal designs

Cost functional

$$J(\mathfrak{a}, u) \;:=\; \int_\Omega g(x)u(x)dx + \frac{1}{2}\|\mathfrak{a}\|^2_{L^2(\Omega)}$$

State equation (weak form):

$$B_{\delta,\mathfrak{A}}(u_\delta, v) \;=\; \langle g, v\rangle, \text{ for all } \; v \in X_0(\Omega_\delta).$$

Theorem (Existence of nonlocal optimal design)

Let $\delta > 0$ be fixed. There exists a pair $(\overline{\mathfrak{a}_\delta}, \overline{u_\delta})$ minimizing

$$J(\overline{\mathfrak{a}_\delta}, \overline{u_\delta}) \;=\; \min_{\mathfrak{a}\in\mathcal{H}, u\in X_0} J(\mathfrak{a}, u),$$

over pairs that satisfy the state equation.

NOTE: Solutions to this problem are not necessarily unique!

Variational convergence as $\delta \to 0^+$

The local problem is to minimize

$$J(\mathfrak{a}, u) \;=\; \int_\Omega g(x)u(x) + \frac{1}{2}\|\mathfrak{a}\|^2_{L^2(\Omega)}$$

over pairs $(\mathfrak{a}, u) \in \mathcal{H} \times H^1_0(\Omega)$ that satisfy

$$B_{0,\mathfrak{a}}(u, v) \;:=\; \frac{1}{n}\int_\Omega \mathfrak{a}(x) \bigtriangledown u(x) \cdot \bigtriangledown v(x) dx \;=\; \langle g, v\rangle \qquad \forall v \in H^1_0(\Omega)$$

Theorem

Suppose $\{(\overline{\mathfrak{a}_\delta}, \overline{u_\delta})\}_{\delta>0}$ *is a family of solutions to the nonlocal optimal design problem. Then there is* $(\bar{\mathfrak{a}}, \bar{u})$ *such that* $\overline{\mathfrak{a}_\delta} \to \bar{\mathfrak{a}}$ *strongly in* $L^2(\Omega)$*,* $\overline{u_\delta} \to \bar{u}$ *strongly in* $L^2(\Omega)$*, and* $(\bar{\mathfrak{a}}, \bar{u})$ *solves the local design problem. In addition, we have that* $\lim_{\delta\to 0^+} J(\overline{\mathfrak{a}_\delta}, \overline{u_\delta}) \;=\; J(\bar{\mathfrak{a}}, \bar{u})$.

Outline

Discrete Framework

- The states are discretized with continuous piecewise linear functions (with zero nonlocal boundary data)
- We use a **variational discretization**, meaning we do not explicitly discretize the design space $\mathcal{H}$

Convergence as $h \to 0^+$

Theorem

Let $\delta \geq 0$ be fixed, and let $\{(\overline{\mathfrak{a}_{\delta,h}}, \overline{u_{\delta,h}})\}_{h>0}$ solve the discrete optimal design problem. Then there exists a sub-sequence of mesh indices and $\overline{\mathfrak{a}_\delta} \in \mathcal{H}$ such that $\overline{\mathfrak{a}_{\delta,h}} \rightharpoonup \overline{\mathfrak{a}_\delta}$ weak- in $L^\infty(\Omega)$. If we denote $\overline{u_{\delta,h}} := \mathcal{L}^{-1}_{\delta,\overline{\mathfrak{A}_{\delta,h}},h} g$ and $\overline{u_\delta} := \mathcal{L}^{-1}_{\delta,\overline{\mathfrak{A}_\delta}} g$, then:*

1. $(\overline{\mathfrak{a}_\delta}, \overline{u_\delta})$ *solves the continuous optimal design problem;*
2. $\overline{u_{\delta,h}} \to \overline{u_\delta}$ *strongly in* $X(\Omega_\delta)$*, and* $\overline{\mathfrak{a}_{\delta,h}} \to \overline{\mathfrak{a}_\delta}$ *strongly in* $L^2(\Omega)$
3. $\lim_{h\to 0^+} J(\overline{\mathfrak{a}_{\delta,h}}, \overline{u_{\delta,h}}) \;=\; J(\overline{\mathfrak{a}_\delta}, \overline{u_\delta})$

Outline

What is asymptotic compatibility?

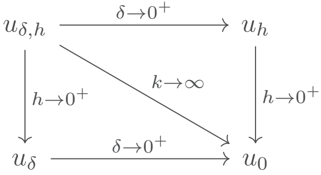
- Introduced by X. Tian and Q. Du (2014)
- Originally developed for linear, nonlocal state equations

$$\mathcal{L}_{\delta,h} u_{\delta,h} = f$$

- Guarantees unconditional convergence of approximations in both discretization and horizon parameters

Definition (Asymptotic Compatibility)

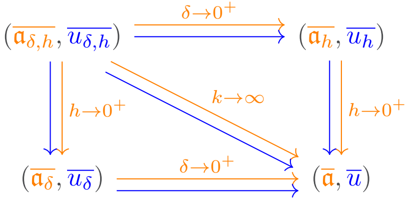
Given fixed data f in a Hilbert Space, the family of solutions $\{u_{\delta,h}\}_{\delta,h>0}$ is **asymptotically compatible** in $\delta, h > 0$ if for any sequences $\{\delta_k\}_{k=1}^{\infty}, \{h_k\}_{k=1}^{\infty}$ with $\delta_k, h_k \to 0$, we have that $u_{\delta_k,h_k} \to u_0$ strongly in some Hilbert space norm, where u_0 is the solution to a local, continuous problem.



Asymptotic Compatibility

Definition (Asymptotic Compatibility for Optimal Design)

We say that a family of nonlocal discrete optimal design problems is **asymptotically compatible** if for any family of solutions $\{(\overline{\mathfrak{a}_{\delta,h}}, \overline{u_{\delta,h}})\}_{h>0,\delta>0}$ and any sequences $\{\delta_k\}_{k=1}^{\infty}, \{h_k\}_{k=1}^{\infty}$ with $\delta_k, h_k \to 0$, there exists a subsequence for which $\overline{\mathfrak{a}_{\delta_k,h_k}} \to \overline{\mathfrak{a}}$ strongly in $L^2(\Omega)$, and $\overline{u_{\delta_k,h_k}} \to \overline{u}$ strongly in $L^2(\Omega)$, an pair solving the local continuous design problem.



NOTE: The solution $(\overline{\mathfrak{a}}, \overline{u})$ reached depends on the choice of sub-sequence!

Theorem

Our family of discrete optimal design problems is asymptotically compatible as $\delta, h \to 0^+$, *and* $\lim_{k\to\infty} J(\overline{\mathfrak{a}_{\delta_k,h_k}}, \overline{u_{\delta_k,h_k}}) = J(\overline{\mathfrak{a}}, \overline{u})$.

Proof of asymptotic compatibility

Pick sequences $\{\delta_k\}_{k=1}^{\infty}$, $\{h_k\}_{k=1}^{\infty}$, let $\overline{u_k} := \overline{u_{\delta_k,h_k}}$, $\overline{\mathfrak{a}_k} := \overline{\mathfrak{a}_{\delta_k,h_k}}$. There exists $\overline{\mathfrak{a}} \in \mathcal{H}$ so that $\overline{\mathfrak{a}_k} \overset{*}{\rightharpoonup} \overline{\mathfrak{a}}$ in weak-* $L^{\infty}(\Omega)$, let $\overline{u}$ be local state corresponding to $\overline{\mathfrak{a}}$

Step 1: Show that $\text{liminf}_{k\to\infty} J(\overline{\mathfrak{a}_k}, \overline{u_k}) \;\geq\; J(\overline{\mathfrak{a}}, \overline{u})$
Let $\mathcal{E}^{\delta_k}_{\overline{\mathfrak{A}_k}}$ denote nonlocal energy, $\mathcal{E}^{\text{loc}}_{\overline{\mathfrak{a}}}$ denote local energy, use the identities

$$\mathcal{E}^{\delta_k}_{\overline{\mathfrak{A}_k}}(\overline{u_k}) \;=\; -\frac{1}{2} B_{\delta_k,\overline{\mathfrak{A}_k}}(\overline{u_k}, \overline{u_k}) \quad \textbf{AND} \quad \mathcal{E}^{\text{loc}}_{\overline{\mathfrak{a}}}(\overline{u}) \;=\; -\frac{1}{2} B_{0,\overline{\mathfrak{a}}}(\overline{u}, \overline{u})$$

to show that

$$\text{liminf}_{k\to\infty} \int_{\Omega} g(x)\overline{u_k}(x)dx \;\geq\; \int_{\Omega} g(x)\overline{u}(x)dx$$

Step 2: Show that $\text{limsup}_{k\to\infty} J(\overline{\mathfrak{a}_k}, \overline{u_k}) \;\leq\; J(\overline{\mathfrak{a}}, \overline{u})$
Let $\widetilde{u_k} \in X_{\delta_k,h_k}$ denote the Ritz projection associated with $\overline{\mathfrak{a}}$, then

$$J(\overline{\mathfrak{a}_k}, \overline{u_k}) \;\leq\; J(\overline{\mathfrak{a}}, \widetilde{u_k})$$

and send $k \to \infty$.
NOTE: The use of $\overline{\mathfrak{a}}$ as a test function is allowed because we are using a variational discretization.

Proof of asymptotic compatibility (continued)

Step 3: Show that $(\overline{\mathfrak{a}}, \overline{u})$ solves the local design problem
Analogous to Step 2

Step 4: Show that $\overline{u_k} \to \overline{u}$ strongly in $L^2(\Omega)$
Using Steps 1-2 we have

$$\lim_{k\to\infty} \int_\Omega g(x)\overline{u_k}(x)dx \;=\; \int_\Omega g(x)\overline{u}(x)dx$$

and then use the state equations to prove

$$\lim_{k\to\infty} \|\overline{u_k} - \overline{u}\|^2_{L^2(\Omega)} \lesssim \; \lim_{k\to\infty} B_{\delta_k,\overline{\mathfrak{A}_k}}(\overline{u_k} - \overline{u}, \overline{u_k} - \overline{u}) \;=\; 0$$

Step 5: Improve coefficient convergence to $\overline{\mathfrak{a}_k} \to \overline{\mathfrak{a}}$ strongly in $L^2(\Omega)$
Due to Steps 1-2 we get

$$\lim_{k\to\infty} \int_\Omega |\overline{\mathfrak{a}_k}(x)|^2 dx \;=\; \int_\Omega |\overline{\mathfrak{a}}(x)|^2 dx$$

Outline

Necessary changes

- Nonlocal bi-linear form: $B_{\delta,\mathfrak{A}}(u,v) := \iint_{\mathcal{D}_\delta} \mathfrak{A}(x,y)\frac{k_\delta(x-y)}{|x-y|^2} Du(x,y)Dv(x,y)dxdy$ where $Du(x,y) = (u(x)-u(y))\cdot\frac{x-y}{|x-y|}$ (here $u:\Omega_\delta \to \mathbb{R}^n$)
- Nonlocal function space: $X(\Omega_\delta;\mathbb{R}^n) := \{u \in L^2(\Omega;\mathbb{R}^n) \mid B_{\delta,\mathfrak{A}}(u,u) < \infty\}$
- Local bi-linear form $B_{0,\mathfrak{a}}(u,v) := \frac{1}{n(n+2)}\int_\Omega \mathfrak{a}(x)(2\langle \mathsf{Sym}(\bigtriangledown u(x)), \mathsf{Sym}(\bigtriangledown v(x))\rangle_F + \mathsf{div}(u(x))\mathsf{div}(v(x)))dx$
- Local function space: $H^1(\Omega;\mathbb{R}^n) := \{u \in L^2(\Omega;\mathbb{R}^n) \mid B_{0,\mathfrak{a}}(u,u) < \infty\}$

NOTE: Same class of design coefficients!

Summary and concluding remarks

- Showed existence of minimizers
- Considered variational convergence as $\delta \to 0^+$
- Discretized via FEM, $h \to 0^+$
- Studied simultaneous limit as $\delta, h \to 0^+$

Questions??

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