An Optimal Control Problem in Peridynamics: Analysis and Discretization

Joshua Siktar
jsiktar@vols.utk.edu

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THE UNIVERSITY OF TENNESSEE KNOXVILLE
DEPARTMENT OF MATHEMATICS

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Problem statement

Find $(\overline{u_\delta}, \overline{g_\delta}) \in X_0 \times Z_{\text{ad}}$ such that

$$I(\overline{u_\delta}, \overline{g_\delta}) \; = \; \min_{g_\delta \in L^2, u_\delta \in X_0} \left\{ \int_\Omega F(x, u_\delta(x))dx + \frac{\eta}{2} \int_\Omega \Lambda(x)|g(x)|^2 dx \right\},$$

over pairs $(u_\delta, g_\delta) \in X_0 \times Z_{\text{ad}}$ that satisfy some state equation

$$B_\delta(u_\delta, w_\delta) \; = \; \int_\Omega g_\delta(x) \cdot w_\delta(x), \; \forall w_\delta \in X_0.$$

where $\delta \geq 0$ is the degree of nonlocality. Here $\overline{g_\delta}$ is an external force and $\overline{u_\delta}$ represents the displacement

Outline

Motivation and origins

For our problem the nonlocal operator is

$$\mathcal{L}_\delta u(x) \; = \; \frac{1}{2}\int_{\Omega_\delta} \mathfrak{A}(x,y) k_\delta(|x-y|) \frac{Du(x,y)}{|x-y|} \frac{y-x}{|x-y|} dy$$

Nonlocal equations [or systems] take the form

$$\begin{cases} \mathcal{L}_\delta u \; = \; g, \; x \in \Omega \\ u \; = \; 0, \qquad x \in \Omega_\delta \setminus \Omega \end{cases}$$

Common in solid state mechanics, including peridynamics

Motivation and origins (continued)

Definition (PD)

Peridynamics (PD) is a nonlocal model for elasticity of solids that uses integrals over derivatives, attributed to Stewart A. Silling

Features:

- Exchanges derivatives in continuum models for integrals (helps address crack formation)
- Treats particles as having a bond between them (**bond-based model**)
- Range of interaction parameterized by δ, called **horizon**
- Material parameters represented by $\mathfrak{a}(x)$ (e.g., density)
- Operator is elliptic (not parabolic or hyperbolic)

Sample candidate integrand

Example:

$$F(x, u(x)) = |u(x) - u_{\text{des}}(x)|^2$$

where u_{des} is the optimal shape of the material in space to fit a pre-determined hole as closely as possible

Goals

- Show existence and uniqueness of minimizers (well-posedness)
- Consider variational convergence as $\delta \to 0^+$
- Discretize via FEM (δ fixed, $h \to 0^+$)
- Study simultaneous limit as $\delta, h \to 0^+$ (asymptotic compatibility)
- Implementation/numerical results

Properties of kernels

Kernel sequence $\{k_\delta\}_{\delta>0}$ radial, integrable, non-negative, supported in $B(0,\delta)$, $k_\delta(r)r^{-2}$ is nonincreasing, and

$$\lim_{\delta\to 0^+}\int_{\mathbb{R}^n} k_\delta(\xi)d\xi \;=\; \delta_0$$

Also, for all $\delta > 0$:

$$\int_{\mathbb{R}^n} k_\delta(\xi)d\xi \;=\; 1$$

$$\lim_{t\to\infty}\int_{\mathbb{R}^n\setminus B(0,t)} k_\delta(\xi)d\xi = 0$$

Notation

- Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $\Omega_\delta := \Omega \cup \{x, \text{dist}(x, \partial\Omega) < \delta\}$
- $\Omega_\delta \setminus \Omega$ is nonlocal boundary
- $\mathcal{D}_\delta := (\Omega \times \Omega_\delta) \cup (\Omega_\delta \times \Omega)$
- Projected difference: $Du(x, y) := \frac{(u(x)-u(y))\cdot(x-y)}{|x-y|}$, nonlocal linearized strain (for vector-valued functions)
- Our material coefficient function is

$$\mathfrak{A}(x, y) \; := \; \frac{\mathfrak{a}(x) + \mathfrak{a}(y)}{2},$$

where there exist $a_{\min}, a_{\max} > 0$ so $a_{\min} \leq \mathfrak{a} \leq a_{\max}$ on Ω_δ.

Bi-linear forms

Nonlocal bi-linear form:

$$B_\delta(u,v) \;:=\; \frac{1}{2}\iint_{\mathcal{D}_\delta} \mathfrak{A}(x,y)k_\delta(x-y)\frac{Du(x,y)}{|x-y|}\frac{Dv(x,y)}{|x-y|}dxdy$$

Local bi-linear form:

$$B_0(u,v) \;:=\; C(n)\int_\Omega \mathfrak{a}(x)(2\langle \mathrm{Sym}(\triangledown u), \mathrm{Sym}(\triangledown v)\rangle_F + \mathrm{div}(u)\mathrm{div}(v))dx,$$

with $C(n) = \frac{1}{(n+2)(n+4)}$

Inner products denoted $\langle \cdot, \cdot \rangle_Y$; L^2-inner product denoted $\langle \cdot, \cdot \rangle$

Function spaces and norms

Our function space is based on B_δ :

$$X(\Omega_\delta;\mathbb{R}^n) \;:=\; \{u|_\Omega \in L^2(\Omega;\mathbb{R}^n) \mid B_\delta(u,u) < \infty\}$$

Version with zero nonlocal boundary data:

$$X_0(\Omega_\delta;\mathbb{R}^n) \;:=\; \{u \in X(\Omega_\delta;\mathbb{R}^n) \mid u = 0 \text{ in } \Omega_\delta \setminus \Omega\}$$

Underlying norm on $X(\Omega_\delta;\mathbb{R}^n)$:

$$\|u\|_{X(\Omega_\delta;\mathbb{R}^n)} \;:=\; \|u\|_{L^2(\Omega_\delta;\mathbb{R}^n)} + \frac{1}{2}\iint_{\mathcal{D}_\delta} k_\delta(x-y)\frac{|Du(x,y)|^2}{|x-y|^2}dxdy$$

Cost functional assumptions

$$I(u,g) \ := \ \int_\Omega F(x,u(x))dx + \frac{\eta}{2}\int_\Omega \Lambda(x)|g(x)|^2 dx$$

Here Z_{ad} is a nonempty, closed, convex, and bounded subset of $L^2(\Omega;\mathbb{R}^n)$, taking the form

$$Z_{\mathrm{ad}} = \{z \in L^\infty(\Omega;\mathbb{R}^n) \mid a \preceq z \preceq b\}$$

Here $[a]_i \leq [b]_i$ for all $i \in \{1,2,\ldots,n\}$ with $a = ([a]_1,\ldots,[a]_n)$ and $b = ([b]_1,\ldots,[b]_n)$ being vector fields in $L^2(\Omega;\mathbb{R}^n)$, $\eta \geq 0$, and $\Gamma \in L^\infty(\Omega)$ is strictly positive.

Cost functional assumptions (continued)

The integrand $F : \Omega \times \mathbb{R}^n \to \mathbb{R}$ possesses the following properties:

1. For all $v \in \mathbb{R}^n$ the mapping $x \mapsto F(x, v)$ is measurable;
2. For all $x \in \Omega$ the mapping $v \mapsto F(x, v)$ is continuous and convex;
3. There exist $c_1 > 0$ and $\ell \in L^1(\Omega)$

$$|F(x, v)| \leq \; c_1 |v|^2 + \ell(x)$$

   for all $x \in \Omega$, $v \in \mathbb{R}^n$.

Admissible classes

$$\mathcal{A}^\delta := \{(v, f) \in X_0(\Omega_\delta; \mathbb{R}^n) \times Z_{\text{ad}} \mid \text{v solves state system with right hand side f for any } w \in X_0(\Omega_\delta; \mathbb{R}^n)\}$$

$$\mathcal{A}^{\text{loc}} := \{(v, f) \in H_0^1(\Omega; \mathbb{R}^n) \times Z_{\text{ad}} \mid v \text{ solves local BVP for any } w \in H_0^1(\Omega; \mathbb{R}^n)\}$$

Outline

Structural properties

- $(X(\Omega_\delta; \mathbb{R}^n), \|\cdot\|_X)$ and $(X_0(\Omega_\delta; \mathbb{R}^n), \|\cdot\|_X)$ are Hilbert
- Can extend $u \in X_0(\Omega_\delta; \mathbb{R}^n)$ by zero to any open set $B \supset \Omega_\delta$, including $\mathbb{R}^n$.
- $H^1(\Omega_\delta; \mathbb{R}^n) \hookrightarrow X(\Omega_\delta; \mathbb{R}^n)$
- $X(\Omega_\delta; \mathbb{R}^n) \Subset L^2(\Omega_\delta; \mathbb{R}^n)$

Nonlocal Poincaré-Korn Inequality

Proposition (Mengesha-Du 2014)

There exists a $\delta_0 > 0$ *and a constant* $C(\delta_0) > 0$ *such that for all* $\delta \in (0, \delta_0]$ *and* $u \in X_0(\Omega_\delta; \mathbb{R}^n)$,

$$\|u\|^2_{L^2(\Omega;\mathbb{R}^n)} \leq \; C(\delta_0) \int_{\Omega_\delta} \int_{\Omega_\delta} \frac{k_\delta(x-y)|Du(x,y)|^2}{|x-y|^2} dxdy.$$

Notes:

- The assumption of $k_\delta(r) r^{-2}$ being non-increasing is needed in the proof
- The projected difference vanishes under infinitesimal rigid displacements

State equation is well-posed!

Theorem (Existence and Uniqueness for State Equation)

For any $g_\delta \in Z_{ad}$, there exists a unique $u_\delta \in X_0(\Omega_\delta; \mathbb{R}^n)$ such that the state system

$$B_\delta(u_\delta, w) \; = \; \langle g_\delta, w \rangle$$

is satisfied for all $w \in X_0(\Omega_\delta; \mathbb{R}^n)$. Furthermore, we have the stability estimate

$$\|u_\delta\|_{X(\Omega_\delta; \mathbb{R}^n)} \lesssim \; \|g_\delta\|_{X(\Omega_\delta; \mathbb{R}^n)^*}$$

for some constant independent of δ.

Well-posedness of optimal control problem

Theorem (Well-posedness)

There exists $(\overline{u_\delta}, \overline{g_\delta}) \in \mathcal{A}^\delta$ *minimizing*

$$I(u_\delta, g_\delta) = \int_\Omega F(x, u_\delta(x))dx + \frac{\eta}{2}\int_\Omega \Lambda(x)|g_\delta(x)|^2 dx,$$

where $(\overline{u_\delta}, \overline{g_\delta})$ *solves*

$$B_\delta(u_\delta, v_\delta) = \int_\Omega g_\delta(x) \cdot v_\delta(x) dx \quad \forall v_\delta \in X_0$$

Furthermore, if F *is strictly convex or* $\eta > 0$*, then the minimizer is unique.*

Outline

Needed compactness result

Proposition (Mengesha-Du 2014)

Suppose $\{u_\delta\}_{\delta>0} \subset L^2(\Omega;\mathbb{R}^n)$ *is a bounded family of vector fields such that*

$$\sup_{\delta>0} \int_{\Omega_\delta}\int_{\Omega_\delta} k_\delta(x-y)\frac{|Du_\delta(x,y)|^2}{|x-y|^2}dxdy \; < \; \infty,$$

where $k_\delta(r)r^{-2}$ *is nonincreasing; then the family has compact closure in* $L^2(\Omega;\mathbb{R}^n)$*, and any limit point* u *belongs to* $H^1(\Omega_\delta;\mathbb{R}^n)$

In fact, we can show $u \in H^1_0(\Omega;\mathbb{R}^n)$!

Local and nonlocal energies

Define for $u \in L^2(\Omega; \mathbb{R}^n)$:

$$
\begin{aligned}
E_\delta(u) &:= \iint_{\mathcal{D}_\delta} \mathfrak{A}(x,y) k_\delta(x-y) \frac{|Du(x,y)|^2}{|x-y|^2} dxdy; \\
E_0(u) &:= C(n) \int_\Omega \mathfrak{a}(x)(2\|\mathrm{Sym}(\nabla u(x))\|_F^2 + \mathrm{div}(u(x))^2) dx
\end{aligned}
$$

where $C(n) := \frac{1}{(n+2)(n+4)}$, and $\|\cdot\|_F$ is the Fröbenius norm. Take to be $+\infty$ when not well-defined

Minimization of local functional

Theorem

Suppose $\{(\overline{u_\delta}, \overline{g_\delta})\}_{\delta>0}$ is the family of solutions to the nonlocal control problem. Then, there is $(\overline{u}, \overline{g})$ such that $\overline{u_\delta} \to \overline{u}$ in $L^2(\Omega; \mathbb{R}^n)$ and $\overline{g_\delta} \rightharpoonup \overline{g}$ in $L^2(\Omega; \mathbb{R}^n)$. Moreover, $(\overline{u}, \overline{g})$ solves the local optimal control problem.

- Notice $\{\overline{u_\delta}\}_{\delta>0}$ have bounded semi-norm so compactness gives $\overline{u} \in H_0^1(\Omega; \mathbb{R}^n)$
- Notice $\{\overline{g_\delta}\}_{\delta>0}$ are bounded in $L^2(\Omega; \mathbb{R}^n)$ so reflexivity gives $\overline{g} \in Z_{\text{ad}}$

Need minimization to be preserved in limit!

Γ-Convergence

Theorem

We say that the family $E_\delta : L^2(\Omega;\mathbb{R}^n) \to \mathbb{R} \cup \{+\infty\}$ ***Γ-converges*** *strongly in* $L^2(\Omega;\mathbb{R}^n)$ *to* $E_0 : L^2(\Omega;\mathbb{R}^n) \to \mathbb{R} \cup \{+\infty\}$ *(denoted* $E_\delta \xrightarrow{\Gamma} E_0$*). That is, the following hold:*

i) ***The liminf inequality:*** *Assume* $u_\delta \to u$ *strongly in* $L^2(\Omega;\mathbb{R}^n)$*. Then*

$$E_0(u) \le liminf_{\delta\to 0^+} E_\delta(u_\delta)$$

ii) ***Recovery sequence property:*** *For each* $u \in L^2(\Omega;\mathbb{R}^n)$*, there exists a sequence* $\{u_\delta\}_{\delta>0}$ *where* $u_\delta \to u$ *strongly in* $L^2(\Omega;\mathbb{R}^n)$ *and*

$$limsup_{\delta\to 0^+} E_\delta(u_\delta) \le E_0(u)$$

Convergence of Minimizers for Energies

Corollary

If $\{v_\delta\}_{\delta>0}$ is a sequence of minimizers for $\{E_\delta\}_{\delta>0}$ over $L^2(\Omega;\mathbb{R}^n)$, v is a limit point of this sequence, and $E_\delta \xrightarrow{\Gamma} E_0$, then v is a minimizer of E_0 on $L^2(\Omega;\mathbb{R}^n)$. Finally,

$$\lim_{\delta\to 0^+} E_\delta(\overline{u_\delta}) - \langle \overline{g_\delta}, \overline{u_\delta}\rangle \;=\; E_0(\overline{u}) - \langle \overline{g}, \overline{u}\rangle$$

Tying together optimal control

Final step: Show $(\overline{u}, \overline{g}) \in \mathcal{A}^{\text{loc}}$ solves local control problem
For any $(v, f) \in \mathcal{A}^{\text{loc}}$,

$$I(\overline{u}, \overline{g}) \;\leq\; \lim_{\delta \to 0^+} I(\overline{u_\delta}, \overline{g_\delta}) \;\leq\; \lim_{\delta \to 0^+} I(v_\delta, f) \;\leq\; I(v, f),$$

where (v_δ, f) are solutions to the nonlocal state equation.
NOTE: Can improve convergence of controls to strong L^2 using a projection formula!

$$\overline{g_\delta}(x) \;=\; \mathbb{P}_{Z_{\text{ad}}}\left(-\frac{1}{\lambda}\overline{p_\delta}(x)\right),$$

$$\overline{g}(x) \;=\; \mathbb{P}_{Z_{\text{ad}}}\left(-\frac{1}{\lambda}\overline{p}(x)\right),$$

where $\overline{p_\delta}, \overline{p}$ are the optimal adjoints.

Outline

Nonlocal discrete problem statement

Find $(\overline{u_{\delta,h}}, \overline{g_{\delta,h}}) \in X_{\delta,h} \times Z_h$ such that

$$I(\overline{u_{\delta,h}}, \overline{g_{\delta,h}}) = \min_{u_{\delta,h} \in X_{\delta,h},\ g_{\delta,h} \in Z_h} I(u_{\delta,h},\ g_{\delta,h}),$$

over pairs $(u_{\delta,h}, g_{\delta,h}) \in X_{\delta,h} \times Z_h$ that satisfy

$$B_\delta(u_{\delta,h}, v_{\delta,h}) \ = \ \langle g_{\delta,h}, v_{\delta,h} \rangle, \qquad \forall v_{\delta,h} \in X_{\delta,h}.$$

Recap:

$$I(u_{\delta,h}, g_{\delta,h}) \ := \ \int_\Omega F(x, u_{\delta,h}(x))dx + \frac{\eta}{2} \|g_{\delta,h}\|^2_{L^2(\Omega;\mathbb{R}^n)}$$

Henceforth assume $\Gamma \equiv 1$

Local discrete problem statement

Find $(\overline{u_h}, \overline{g_h}) \in X_h \times Z_h$ such that

$$I(\overline{u_h}, \overline{g_h}) \;=\; \min_{u_h \in X_h,\; g_h \in Z_h} I(u_h, g_h),$$

over pairs $(u_h, g_h) \in X_h \times Z_h$ that satisfy

$$B_0(u_h, v_h) \;=\; \langle g_h, v_h \rangle, \qquad \forall v_h \in Z_h.$$

Notation

- Mesh family: $\{\mathscr{T}_h\}_{h>0}$ (discretizing Ω_δ) shape-regular and quasi-uniform
- Piecewise polynomials of degree m (with respect to our mesh):

$$\mathcal{P}_m(T;\mathbb{R}^n) := \left\{ \sum_{\alpha\in\mathbb{N}_0^n \,:\, \sum_{i=1}^n \alpha_i \le m} v_\alpha x_1^{\alpha_1}\cdots x_n^{\alpha_n} \;\middle|\; v_\alpha \in \mathbb{R}^n,\ (x_i)_{i=1}^n \in T \right\}$$

- Discretized state space: $X_{\delta,h} := X_h := \{w_h \in C^0(\overline{\Omega_\delta};\mathbb{R}^n) \mid w_h|_T \in \mathcal{P}_1(T;\mathbb{R}^n)\ \forall T \in \mathscr{T}_h, w_h = 0 \text{ on } \Omega_\delta \setminus \Omega\}$
- Discretized control space: $Z_h := \{z_h|_T \in \mathcal{P}_0(T;\mathbb{R}^n)\ \forall T \in \mathscr{T}_h\}$
- $\Pi_0 : Z_{\text{ad}} \to Z_h$ is piecewise constant projection by averages on each triangle

Notation (continued)

Nonlocal discrete control space: $(X_{\delta,h}, \|\cdot\|_X)$
Local discrete control space $(X_h, \|\cdot\|_{H^1})$

$$\mathcal{A}_h^{\delta} := \{(w_{\delta,h}, f_{\delta,h}) \in X_{\delta,h} \times Z_h \mid w_{\delta,h} \text{ solves system with right-hand side } f_{\delta,h} \text{ for any } v_{\delta,h} \in X_{\delta,h}\}$$

$$\mathcal{A}_h^{\text{loc}} := \{(w_h, f_h) \in X_h \times Z_h \mid w_h \text{ solves local system with right-hand side } f_h \text{ for any } v_h \in X_h\}$$

Optimality conditions preliminaries

Reduced cost functional:

$$j_\delta(g_\delta) \ := \ \int_\Omega F(x, S_\delta g_\delta(x))dx + \frac{\eta}{2}\|g_\delta\|^2_{L^2(\Omega;\mathbb{R}^n)}$$

Assume now that

$$F(x, v) := \frac{1}{2}|v|^2.$$

First-order necessary condition:

$$\langle j'_\delta(\overline{g_\delta}), \gamma_z - \overline{g_\delta}\rangle \ \geq \ 0 \qquad \forall \gamma_z \in Z_{\mathrm{ad}}$$

Optimality conditions

Nonlocal continuous optimality conditions (including adjoint)

$$
\begin{aligned}
\langle \overline{p_\delta} + \eta \overline{g_\delta}, \gamma_z - \overline{g_\delta} \rangle &\geq 0, \quad \forall \gamma_z \in Z_{\mathrm{ad}} \\
\overline{p_\delta} = S_\delta^* F_u(\cdot, \overline{u_\delta}) &= S_\delta F_u(\cdot, \overline{u_\delta}) \\
\overline{u_\delta} &= S_\delta \overline{g_\delta}.
\end{aligned}
$$

Projection formula:

$$
\overline{g_\delta}(x) = \mathbb{P}_{Z_{\mathrm{ad}}} \left(-\frac{1}{\lambda} \overline{p_\delta}(x) \right).
$$

NOTE: No second-order optimality conditions needed (strict convexity!)

Optimality conditions (discretized)

Nonlocal discrete optimality conditions (including adjoint)

$$\begin{aligned}
\langle \overline{p_{\delta,h}} + \eta \overline{g_{\delta,h}}, \gamma_h - \overline{g_{\delta,h}} \rangle &\geq 0, \quad \forall \gamma_h \in Z_{\text{ad}} \cap Z_h \\
\overline{p_{\delta,h}} = S^*_{\delta,h} F_u(\cdot, \overline{u_{\delta,h}}) &= S_{\delta,h} F_u(\cdot, \overline{u_{\delta,h}}) \\
\overline{u_{\delta,h}} &= S_{\delta,h} \overline{g_{\delta,h}}.
\end{aligned}$$

Projection formula:

$$\overline{g_{\delta,h}}(x) = \mathbb{P}_{Z_{\text{ad}}} \left(-\frac{1}{\eta} \Pi_0 \overline{p_{\delta,h}}(x) \right)$$

Intermediary functions

By Lax-Milgram, we may define $\widehat{u_\delta}, \widehat{p_\delta} \in X_0(\Omega_\delta; \mathbb{R}^n)$ such that

$$B_\delta(\widehat{u_\delta}, v_\delta) = \langle \overline{g_{\delta,h}}, v_\delta \rangle \quad \forall v_\delta \in X_0(\Omega_\delta; \mathbb{R}^n);$$

$$B_\delta(v_\delta, \widehat{p_\delta}) = \langle v_\delta, \overline{u_{\delta,h}} \rangle \quad \forall v_\delta \in X_0(\Omega_\delta; \mathbb{R}^n);$$

also define $\widehat{u_h}, \widehat{p_h} \in H_0^1(\Omega; \mathbb{R}^n)$ such that

$$B_0(\widehat{u_h}, v) = \langle \overline{g_h}, v \rangle \quad \forall v \in H_0^1(\Omega; \mathbb{R}^n);$$

$$B_0(v, \widehat{p_h}) = \langle v, \widehat{u_h} \rangle \quad \forall v \in H_0^1(\Omega; \mathbb{R}^n).$$

State and adjoint error estimate

Theorem (State and Adjoint Error Estimates)

Suppose that $(\overline{u_{\delta,h}}, \overline{g_{\delta,h}})$ *is the solution to the nonlocal discrete problem,* $\overline{p_{\delta,h}}$ *solves the discrete adjoint equation given* $\overline{u_{\delta,h}}$*;* $(\overline{u_\delta}, \overline{g_\delta})$ *is the solution to the nonlocal continuous problem; and* $\overline{p_\delta}$ *solves the continuous adjoint equation given* $\overline{u_\delta}$*. Then*

$$\|\overline{u_\delta} - \overline{u_{\delta,h}}\|_{X(\Omega_\delta;\mathbb{R}^n)} \lesssim \inf_{v_{\delta,h}\in X_{\delta,h}} \|\widehat{u_\delta} - v_{\delta,h}\|_{X(\Omega_\delta;\mathbb{R}^n)} + \|\overline{g_\delta} - \overline{g_{\delta,h}}\|_{L^2(\Omega;\mathbb{R}^n)};$$

$$\|\overline{p_\delta} - \overline{p_{\delta,h}}\|_{X(\Omega_\delta;\mathbb{R}^n)} \lesssim \inf_{v_{\delta,h}\in X_{\delta,h}} \|\widehat{p_\delta} - v_{\delta,h}\|_{X(\Omega_\delta;\mathbb{R}^n)} + \inf_{v_{\delta,h}\in X_{\delta,h}} \|\widehat{u_\delta} - v_{\delta,h}\|_{X(\Omega_\delta;\mathbb{R}^n)} + \|\overline{g_\delta} - \overline{g_{\delta,h}}\|_{L^2(\Omega;\mathbb{R}^n)}$$

State and adjoint error estimates (continued)

Proof strategy for state error:

- Test with $v_\delta := \overline{u_\delta} - \widehat{u_\delta}$ in state system and intermediary equation, subtract
- Use Hölder Inequality to get $\|\overline{u_\delta} - \widehat{u_\delta}\|_{X(\Omega_\delta;\mathbb{R}^n)} \lesssim \|\overline{g_\delta} - \overline{g_{\delta,h}}\|_{L^2(\Omega;\mathbb{R}^n)}$
- Use Ceá's Lemma on space $X_{\delta,h}$

Note: Adjoint error estimate proven by same strategy

State and adjoint error estimates (local)

Theorem

Let $(\overline{u}, \overline{g})$ denote the solution to the local continuous problem, while $(\overline{u_h}, \overline{g_h})$ is the solution to the local discrete problem. Assume also that $\overline{p}$ denotes the solution to the continuous adjoint problem, while $\overline{p_h}$ solves the discrete adjoint problem. Then,

$$\|\overline{u} - \overline{u_h}\|_{H^1(\Omega;\mathbb{R}^n)} \lesssim \inf_{v_h \in X_h} [\widehat{u}_h - v_h]_{H^1(\Omega;\mathbb{R}^n)} + \|\overline{g} - \overline{g_h}\|_{L^2(\Omega;\mathbb{R}^n)};$$

$$\|\overline{p} - \overline{p_h}\|_{H^1(\Omega;\mathbb{R}^n)} \lesssim \inf_{v_{\delta,h} \in X_h} [\widehat{p}_h - v_h]_{H^1(\Omega;\mathbb{R}^n)} + \inf_{v_h \in X_h} [\widehat{u}_h - v_h]_{H^1(\Omega;\mathbb{R}^n)} + \|\overline{g} - \overline{g_h}\|_{L^2(\Omega;\mathbb{R}^n)}.$$

Control convergence

Theorem (Convergence of Controls)

Assume that $\overline{g_\delta}$ is the optimal control associated with the nonlocal continuous problem, and $\overline{g_{\delta,h}}$ be the discrete optimal control. Then we have the convergence

$$\|\overline{g_\delta} - \overline{g_{\delta,h}}\|_{L^2(\Omega;\mathbb{R}^n)} \lesssim \omega(h) + \inf_{v_{\delta,h}\in X_{\delta,h}} [\overline{u_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)} + \inf_{v_{\delta,h}\in X_{\delta,h}} [\overline{p_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)}.$$

Proof

Nonlocal problem convergence: summary

Corollary (Full Norm Solution Convergence)

In the setting of our problem formulation,

$$\|\overline{u_\delta} - \overline{u_{\delta,h}}\|_{X(\Omega_\delta;\mathbb{R}^n)} \lesssim \omega(h) + \inf_{v_{\delta,h}\in X_{\delta,h}} \|\widehat{u_\delta} - v_{\delta,h}\|_{X(\Omega_\delta;\mathbb{R}^n)} + \inf_{v_{\delta,h}\in X_{\delta,h}} [\overline{u_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)} + \inf_{v_{\delta,h}\in X_{\delta,h}} [\overline{p_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)};$$

$$\|\overline{p_\delta} - \overline{p_{\delta,h}}\|_{X(\Omega_\delta;\mathbb{R}^n)} \lesssim \omega(h) + \inf_{v_{\delta,h}\in X_{\delta,h}} \|\widehat{p_\delta} - v_{\delta,h}\|_{X(\Omega_\delta;\mathbb{R}^n)} + \inf_{v_{\delta,h}\in X_{\delta,h}} \|\widehat{u_\delta} - v_{\delta,h}\|_{X(\Omega_\delta;\mathbb{R}^n)} + \inf_{v_{\delta,h}\in X_{\delta,h}} [\overline{u_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)} + \inf_{v_{\delta,h}\in X_{\delta,h}} [\overline{p_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)}.$$

Local convergence results

Theorem

Suppose $(\overline{u}, \overline{g})$ denotes the solution to the local continuous problem, while $(\overline{u_h}, \overline{g_h})$ is the solution to the local discrete problem, and $\overline{p}$ and $\overline{p_h}$ solve the respective adjoint problems. We have the estimates

$$\|\overline{u} - \overline{u_h}\|_{H^1(\Omega;\mathbb{R}^n)} \lesssim \inf_{v_h \in X_h} [\widehat{u} - v_h]_{H^1(\Omega;\mathbb{R}^n)} + \|\overline{g} - \overline{g_h}\|_{L^2(\Omega;\mathbb{R}^n)};$$

$$\|\overline{p} - \overline{p}\|_{H^1(\Omega;\mathbb{R}^n)} \lesssim \inf_{v_h \in X_h} [\widehat{p} - v_h]_{H^1(\Omega;\mathbb{R}^n)} + \inf_{v_h \in X_h} [\widehat{u} - v_h]_{H^1(\Omega;\mathbb{R}^n)} + \|\overline{g} - \overline{g_h}\|_{L^2(\Omega;\mathbb{R}^n)}.$$

$$\|\overline{g} - \overline{g_h}\|_{L^2(\Omega;\mathbb{R}^n)} \lesssim h + \inf_{v_h \in X_h} [\overline{p} - v_h]_{H^1(\Omega;\mathbb{R}^n)} + \inf_{v_h \in X_h} [\overline{u} - v_h]_{H^1(\Omega;\mathbb{R}^n)}.$$

Discrete analogues of convergence

Proposition (Γ-convergence of discrete problems)

We have that $E_\delta \xrightarrow{\Gamma} E_0$ *in the family of spaces* $\{X_{\delta,h}\}_{\delta>0}$ *in the strong* $L^2(\Omega;\mathbb{R}^n)$ *topology.*

Theorem (Discrete Convergence)

Suppose $\{(\overline{u_{\delta,h}}, \overline{g_{\delta,h}})\}_{\delta>0} \in \mathcal{A}_h^\delta$ *is the family of solutions to the nonlocal discrete problem. Then, there is* $(\overline{u_h}, \overline{g_h}) \in \mathcal{A}_h^{loc}$ *such that* $\overline{u_{\delta,h}} \to \overline{u_h}$ *in* $L^2(\Omega;\mathbb{R}^n)$ *and* $\overline{g_{\delta,h}} \to \overline{g_h}$ *in* $L^2(\Omega;\mathbb{R}^n)$*. Moreover,* $(\overline{u_h}, \overline{g_h})$ *solves the local discrete optimal control problem.*

Outline

What is asymptotic compatibility?

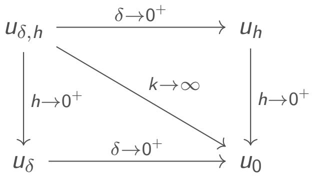
- Introduced by X. Tian and Q. Du (2014)
- Originally developed for linear, non-local state equations

$$\mathcal{L}_{\delta,h} u_{\delta,h} = f$$

- Guarantees unconditional convergence of approximations in both discretization and horizon parameters

Definition (Asymptotic Compatibility)

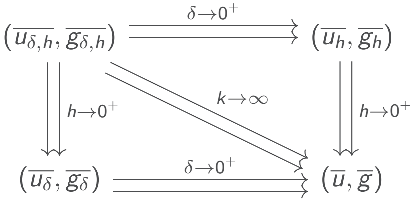
Given fixed data f in a Hilbert Space, the family of solutions $\{u_{\delta,h}\}_{\delta,h>0}$ is **asymptotically compatible** in $\delta, h > 0$ if for any sequences $\{\delta_k\}_{k=1}^{\infty}, \{h_k\}_{k=1}^{\infty}$ with $\delta_k, h_k \to 0$, we have that $u_{\delta_k,h_k} \to u_0$ strongly in some Hilbert space norm, where u_0 is the solution to a local, continuous problem.



What is asymptotic compatibility? (continued)

Definition (Asymptotic Compatibility for Optimal Control)

We say that the family of solutions $\{(\overline{u_{\delta,h}}, \overline{g_{\delta,h}})\}_{h>0,\delta>0}$ to the nonlocal discrete optimal control problem is **asymptotically compatible** in $\delta, h > 0$ if for any sequences $\{\delta_k\}_{k=1}^{\infty}, \{h_k\}_{k=1}^{\infty}$ with $\delta_k, h_k \to 0$, we have that $\overline{g_{\delta_k,h_k}} \to \overline{g}$ strongly in $L^2(\Omega;\mathbb{R}^n)$, and $\overline{u_{\delta_k,h_k}} \to \overline{u}$ strongly in $L^2(\Omega;\mathbb{R}^n)$. Here $(\overline{u}, \overline{g}) \in H_0^1(\Omega;\mathbb{R}^n) \times Z_{\text{ad}}$ denotes the optimal solution for the local continuous problem.



Asymptotic compatibility: result and notation

Theorem

Our family of discrete optimal control problems is asymptotically compatible as $\delta, h \to 0^+$, *and* $\lim_{k\to\infty} I(\overline{u_{\delta_k,h_k}}, \overline{g_{\delta_k,h_k}}) = I(\overline{u}, \overline{g})$.

For convenience denote $(\overline{u_k}, \overline{g_k}, \overline{p_k}) \;:=\; (\overline{u_{\delta_k,h_k}}, \overline{g_{\delta_k,h_k}}, \overline{p_{\delta_k,h_k}})$.

Step 0: Pick sub-sequence of $\{(\overline{u_k}, \overline{g_k}, \overline{p_k})\}_{k=1}^{\infty}$ so there is a limit point $(u_*, g_*, p_*) \in H_0^1(\Omega;\mathbb{R}^n) \times Z_{\text{ad}} \times H_0^1(\Omega;\mathbb{R}^n)$ with convergence in appropriate topology (weak convergence of controls)

Define the family of nonlocal energies $\mathscr{J}^{\delta_k} : X(\Omega_\delta;\mathbb{R}^n) \to \mathbb{R}$ as

$$\mathscr{J}^{\delta_k}(u) \;:=\; \iint_{\mathcal{D}_\delta} \mathfrak{A}(x,y)\frac{k_{\delta_k}(x-y)}{|x-y|^2}|Du(x,y)|^2 dxdy - \langle \overline{g_k}, u\rangle$$

and the local energy $\mathscr{J}^{\text{loc}} : H^1(\Omega;\mathbb{R}^n) \to \mathbb{R}$ as

$$\mathscr{J}^{\text{loc}}(u) \;:=\; \int_\Omega \mathfrak{a}(x)(2\|\text{Sym}(\nabla u(x))\|_F^2 + \text{div}(u(x))^2)dx - \langle g_*, u\rangle$$

Proof of asymptotic compatibility

Step 1: Show that $B_0(u_*, v) = \langle g_*, v \rangle$ for all $\varphi \in H_0^1(\Omega; \mathbb{R}^n)$
Construct a sequence $\{v_k\}_{k=1}^{\infty} \subset X_{\delta_k, h_k}$ so that $v_k \to v$ strongly in $H^1(\Omega; \mathbb{R}^n)$.
Then $B_{\delta_k}(\overline{u_k}, v_k) = \langle \overline{g_k}, v_k \rangle$, or equivalently,

$$\mathscr{J}^{\delta_k}(\overline{u_k}) \leq \mathscr{J}^{\delta_k}(v_k)$$

Then by Γ-convergence,

$$\mathscr{J}^{\text{loc}}(u_*) \leq \mathscr{J}^{\text{loc}}(v).$$

Step 2: Show that $B_0(v, p_*) = \langle u_*, v \rangle$ for all $v \in H_0^1(\Omega; \mathbb{R}^n)$
Strategy is identical to Step 1!

Proof of asymptotic compatibility (continued)

Step 3: Show that $g_*(x) \ = \ \mathbb{P}_{Z_{\text{ad}}}\left(-\frac{1}{\lambda} p_*(x)\right)$
Recall that $\overline{g_k}(x) \ = \ \mathbb{P}_{Z_{\text{ad}}}\left(-\frac{1}{\lambda}\Pi_0\overline{p_k}(x)\right)$, show $\Pi_0\overline{p_k} \to p_*$ strongly in $L^2(\Omega;\mathbb{R}^n)$

Step 4: Unraveling

- Steps 1-3 and uniqueness of solutions to optimality system give $\overline{u} = u_*$, $\overline{g} = g_*$, and $\overline{p} = p_*$
- This is the limit point reached for any sub-sequence of original sequence
- Entire sequence of triples $\{(\overline{u_k}, \overline{g_k}, \overline{p_k})\}_{k=1}^{\infty}$ converges to $(\overline{u}, \overline{g}, \overline{p})$

Step 5: Strong convergence of controls
Use the Lipschitz property of the projection and the estimate

$$\|\overline{g_k} - \overline{g}\|_{L^2(\Omega;\mathbb{R}^n)} \lesssim \ \|\overline{p} - \Pi_0\overline{p}\|_{L^2(\Omega;\mathbb{R}^n)} + \|\Pi_0\overline{p} - \Pi_0\overline{p_k}\|_{L^2(\Omega;\mathbb{R}^n)}$$

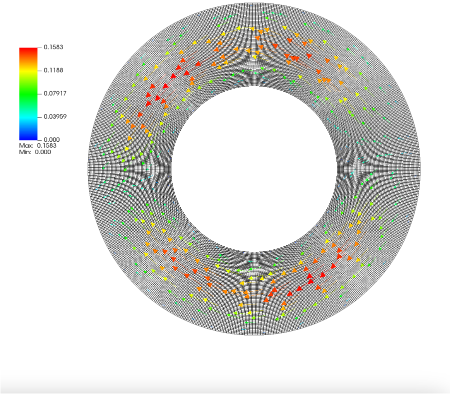
Outline

Local control problem numerical test

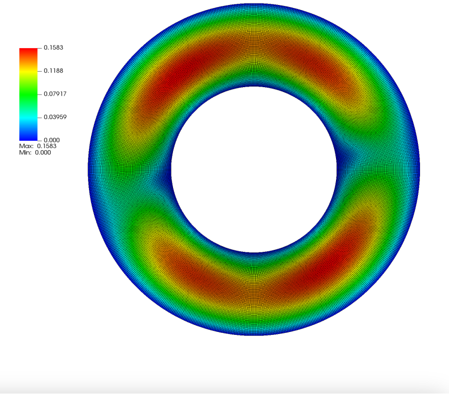
Problem parameters: Tolerance 10^{-12}, $n = 2$, $\eta = 0.01$, domain $\Omega := B(0,2) \setminus B(0,1)$
Desired state: $(u_{\text{des}})_1(x,y) := -y\sin(2\theta)\cos(3\pi r)(r-1)(r-2)$ and $(u_{\text{des}})_2(x,y) := x\sin(2\theta)\cos(3\pi r)(r-1)(r-2)$, where $r := \sqrt{x^2+y^2}$ and $\theta := \tan^{-1}\left(\frac{y}{x}\right)$

h	$\|\overline{u_h}\|_{L^2(\Omega;\mathbb{R}^2)}$	$\|\overline{u_h}\|_{H^1(\Omega;\mathbb{R}^2)}$	$\|\overline{g_h}\|_{L^2(\Omega;\mathbb{R}^2)}$	Number of iterations
2^{-2}	0.0091437	0.0440521	0.312603	63
2^{-3}	0.0116019	0.0599568	0.388542	91
2^{-4}	0.0124453	0.0650956	0.412412	102
2^{-5}	0.0126701	0.0664565	0.418661	106
2^{-6}	0.0127272	0.0668017	0.420242	107
h	$\|\overline{p_h}\|_{L^2(\Omega;\mathbb{R}^2)}$	$\|\overline{p_h}\|_{H^1(\Omega;\mathbb{R}^2)}$	$j_h^{\text{loc}}(\overline{g_h})$	$\|\overline{u_h} - u_{\text{des}}\|_{L^2(\Omega;\mathbb{R}^2)}$
2^{-2}	0.00354106	0.0235156	0.0407674	0.283827
2^{-3}	0.00400814	0.0276332	0.0449658	0.297358
2^{-4}	0.00415522	0.0284508	0.0448784	0.296742
2^{-5}	0.00419442	0.0286662	0.0448501	0.296559
2^{-6}	0.00420437	0.0287208	0.0448428	0.296512

Local control problem simulations



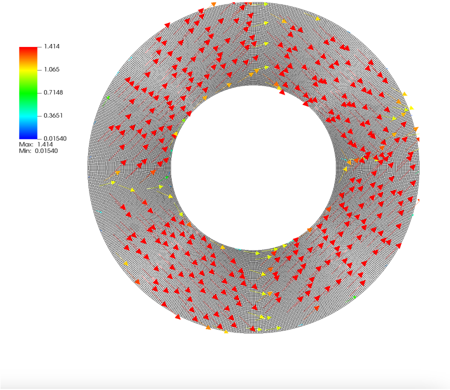
(a) Vector field



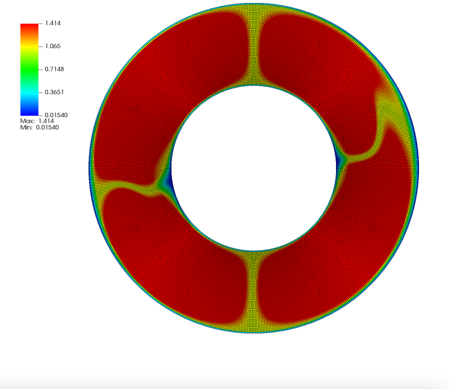
(b) Pseudocolor plot

Figure: Local elasticity state on annular domain, $n = 2$, $\eta = .001$, 6 global refinements, tolerance 10^{-12}

Local control problem simulations (continued)



(a) Vector field



(b) Pseudocolor plot

Figure: Local elasticity control on annular domain, $n = 2$, $\eta = .001$, 6 global refinements, tolerance 10^{-12}

Challenges of nonlocal control code

Local contribution of bi-linear form ($T, T' \in \mathscr{T}_h$):

$$\mathcal{A}_{T,T'}(\phi_i, \phi_j) \; := \; C_{n,s} \int_T \int_{T'} \frac{\chi_{B(0,\delta)}(x-y)}{|x-y|^{n+2s-2}} D\phi_i(x,y) D\phi_j(x,y) dxdy$$

NOTE: Sample kernel: $k_\delta(\xi) \sim \frac{1}{|\xi|^{n+2s-2}}$

Challenges/novelties:

- Domain: Ω and $\Omega_\delta \setminus \Omega$ are separately discretized with rectangular meshes, $\Omega = [0,1]^n$
- Dense stiffness matrix when $\delta >> h$
- Must implement projected differences
- Near-field case has singularity that must be transformed via Duffy transforms
- Far-field case applies to $O(N^2)$ entries of stiffness matrix (away from main diagonal)

NOTE: One advantage from an efficiency advantage is that $\text{supp}(k_\delta) \subset B(0, \delta)$, meaning we can omit calculations for some far-field integrations

Recap and closing

- Showed existence and uniqueness of minimizers
- Considered behavior as $\delta \to 0^+$
- Discretized via FEMs
- Studied simultaneous limit as $\delta, h \to 0^+$ (asymptotic compatibility)

Thank you! Questions?

ArXiV preprint: `https://arxiv.org/pdf/2304.09328.pdf`

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Auxiliary Lemmas

Lemma (Regularity of Control for Fractional-Type Kernels)

Suppose that

$$\frac{c}{|\xi|^{n+2s}} \leq \frac{k_\delta(\xi)}{|\xi|^2} \leq \frac{C}{|\xi|^{n+2s}}$$

holds for all $\xi \in B(0,\delta)$, for some $s \neq \frac{1}{2}$. Then necessarily $\overline{g_\delta} \in X(\Omega_\delta; \mathbb{R}^n)$.

Proof strategy:

- Mengesha-Du 2016 says that $H^s(\Omega_\delta; \mathbb{R}^n) = X(\Omega_\delta; \mathbb{R}^n)$ here
- Use projection formula $\overline{g_\delta}(x) = -\frac{1}{\lambda}\mathbb{P}_{Z_{\text{ad}}}(\overline{p_\delta}(x))$
- This pointwise projection is continuous in H^s semi-norm but not X semi-norm!

Auxiliary Lemmas (continued)

Lemma (Approximation)

If $w \in L^2(\Omega_\delta; \mathbb{R}^n)$, then

$$\|\Pi_0 w - w\|_{L^2(\Omega;\mathbb{R}^n)} \leq \omega(h),$$

where $\Pi_0 : L^2(\Omega_\delta; \mathbb{R}^n) \to Z_h$ denotes the projection of a function onto the piecewise constants with respect to the given mesh. If in fact $w \in X(\Omega_\delta; \mathbb{R}^n)$ and k_δ satisfies the fractional inequality on $B(0, \delta)$ for some $s \neq \frac{1}{2}$, then

$$\|\Pi_0 w - w\|_{L^2(\Omega;\mathbb{R}^n)} \lesssim h^s \|w\|_{X(\Omega_\delta;\mathbb{R}^n)}.$$

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Control Convergence (continued)

Galerkin Approximations:

$q_{\delta,h} \in X_{\delta,h}$ be the Galerkin approximation to $\overline{p_\delta}$, i.e., the solution of

$$B_\delta(v_{\delta,h}, q_{\delta,h}) \;=\; \langle \overline{u_\delta}, v_{\delta,h} \rangle \qquad \forall v_{\delta,h} \in X_{\delta,h}.$$

$U_{\delta,h} \in X_{\delta,h}$ for $\overline{u_\delta}$:

$$B_\delta(U_{\delta,h}, v_{\delta,h}) \;=\; \langle \overline{g_\delta}, v_{\delta,h} \rangle \qquad \forall v_{\delta,h} \in X_{\delta,h}.$$

$r_{\delta,h} \in X_{\delta,h}$ solves

$$B_\delta(v_{\delta,h}, r_{\delta,h}) \;=\; \langle U_{\delta,h}, v_{\delta,h} \rangle \qquad \forall v_{\delta,h} \in X_{\delta,h}.$$

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Control Convergence (continued)

Let $I_1 := \langle \overline{p_\delta} - \overline{p_{\delta,h}}, \overline{g_{\delta,h}} - \overline{g_\delta} \rangle$ and $I_2 := \langle \overline{p_{\delta,h}} + \eta \overline{g_{\delta,h}}, \Pi_0 \overline{g_\delta} - \overline{g_\delta} \rangle$. Using optimality conditions gives

$$\eta \| \overline{g_\delta} - \overline{g_{\delta,h}} \|^2_{L^2(\Omega;\mathbb{R}^n)} \leq \; I_1 + I_2.$$

$$I_1 \;=\; \langle \overline{p_\delta} - q_{\delta,h}, \overline{g_{\delta,h}} - \overline{g_\delta} \rangle + \langle q_{\delta,h} - r_{\delta,h}, \overline{g_{\delta,h}} - \overline{g_\delta} \rangle + \langle r_{\delta,h} - \overline{p_{\delta,h}}, \overline{g_{\delta,h}} - \overline{g_\delta} \rangle \;=: I_{1,1} + I_{1,2} + I_{1,3}.$$

By use of Galerkin approximations, find that $I_{1,3} \leq 0$ and

$$I_{1,1} \;\lesssim\; \| \overline{g_{\delta,h}} - \overline{g_\delta} \|_{L^2(\Omega;\mathbb{R}^n)} \inf_{v_{\delta,h} \in X_{\delta,h}} [\overline{p_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)};$$

$$I_{1,2} \;\lesssim\; \| \overline{g_{\delta,h}} - \overline{g_\delta} \|_{L^2(\Omega;\mathbb{R}^n)} \inf_{v_{\delta,h} \in X_{\delta,h}} [\overline{u_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)}.$$

Control Convergence (continued)

By Young's Inequality,

$$I_1 \le \frac{\eta}{3}\|\overline{g_{\delta,h}} - \overline{g_\delta}\|^2_{L^2(\Omega;\mathbb{R}^n)} + C\left(\inf_{v_{\delta,h}\in X_{\delta,h}}[\overline{u_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)}\right)^2 + C\left(\inf_{v_{\delta,h}\in X_{\delta,h}}[\overline{p_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)}\right)^2.$$

Control Convergence (continued)

Now we write I_2 as

$$\langle\overline{p_{\delta,h}}+\eta\overline{g_{\delta,h}},\Pi_0\overline{g_\delta}-\overline{g_\delta}\rangle \;=\; \langle\overline{p_\delta}+\eta\overline{g_\delta},\Pi_0\overline{g_\delta}-\overline{g_\delta}\rangle+\lambda\langle\overline{g_{\delta,h}}-\overline{g_\delta},\Pi_0\overline{g_\delta}-\overline{g_\delta}\rangle+$$
$$\langle\overline{p_{\delta,h}}-r_{\delta,h},\Pi_0\overline{g_\delta}-\overline{g_\delta}\rangle+\langle r_{\delta,h}-q_{\delta,h},\Pi_0\overline{g_\delta}-\overline{g_\delta}\rangle+\langle q_{\delta,h}-\overline{p_\delta},\Pi_0\overline{g_\delta}-\overline{g_\delta}\rangle \;=:$$
$$I_{2,1}+I_{2,2}+I_{2,3}+I_{2,4}+I_{2,5}.$$

Control Convergence (continued)

Use that $\Pi_0(\overline{p_\delta} + \eta\overline{g_\delta}) = 0$ to estimate $I_{2,1}$ as

$$I_{2,1} \;\leq\; \omega(h).$$

For $I_{2,2}$, use Cauchy and stability:

$$I_{2,2} \;\leq\; \frac{\eta}{3}\|\overline{g_{\delta,h}} - \overline{g_\delta}\|^2_{L^2(\Omega;\mathbb{R}^n)} + \omega(h),$$

Control Convergence (continued)

For $I_{2,3}$, use Galerkin approximations and the stabilty:

$$I_{2,3} \;\leq\; \omega(h)^2 + \left(\inf_{v_{\delta,h} \in X_{\delta,h}} [\overline{u_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)} \right)^2$$

For $I_{2,4}$ and $I_{2,5}$, use Ceá's lemma and Cauchy:

$$I_{2,4} \;\leq\; \omega(h)^2 + \left(\inf_{v_{\delta,h} \in X_{\delta,h}} [\overline{u_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)} \right)^2 ;$$

$$I_{2,5} \;\leq\; \omega(h)^2 + \left(\inf_{v_{\delta,h} \in X_{\delta,h}} [\overline{p_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)} \right)^2 .$$

Control Convergence (continued)

Use Young's Inequality and combine all the estimates to get

$$\frac{\eta}{3}\|\overline{g_\delta} - \overline{g_{\delta,h}}\|^2_{L^2(\Omega;\mathbb{R}^n)} \lesssim \omega(h)^2 + \left(\inf_{v_{\delta,h}\in X_{\delta,h}} [\overline{u_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)}\right)^2 + \left(\inf_{v_{\delta,h}\in X_{\delta,h}} [\overline{p_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)}\right)^2 .$$

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