Asymptotically Compatible Schemes for Discretization of Nonlocal Problems

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Paper Highlighted

X. Tian, Q. Du (2014). Asymptotically Compatible Schemes and Applications to Robust Discretization of Nonlocal Models. SIAM J. Numer. Anal. **52**(4), 1641-1665.

Areas of emphasis

- Functional analysis

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- Functional analysis
- Nonlocal modeling/peridynamics (PD)
- Numerical analysis

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- Construct suitable finite element spaces for Galerkin approximations

Outline of Talk

- Functional Analytic Framework

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- Numerics

Setup

- $\{(\mathcal{T}_\sigma, ||\cdot||_\sigma)\}_{\sigma\geq 0}$ decreasing family of Hilbert spaces

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- $\mathcal{T}_{-\sigma} := \mathcal{T}_\sigma^*$

Assumptions

Definition

Uniform Embedding: There exist $M_1, M_2 > 0$ such that

$$M_1 ||u||_{\mathcal{T}_0} \leq ||u||_{\mathcal{T}_\sigma}, \; \forall u \in \mathcal{T}_\sigma$$

$$||u||_{\mathcal{T}_\sigma} \leq M_2 ||u||_{\mathcal{T}_\infty}, \; \forall u \in \mathcal{T}_\infty$$

Definition

Asymptotic Compact Embedding: For $\{u_n\}_{n>0}$ with $C > 0$ s.t. $||u_n||_{\mathcal{T}_n} \leq C$, $\{u_n\}_{n>0}$ is relatively compact in $\mathcal{T}_0$.

Assumptions (continued)

For $\sigma \in [0, \infty]$, $a_\sigma : \mathcal{T}_\sigma \times \mathcal{T}_\sigma \to \mathbb{R}$ a symmetric bilinear form:

Definition

Coercive: there exists $C_1 > 0$ s.t.

$$a_\sigma(u, u) \geq C_1 ||u||^2_{\mathcal{T}_\sigma}, \ \forall u \in \mathcal{T}_\sigma$$

Definition

Bounded: there exists $C_2 > 0$ s.t.

$$a_\sigma(u, v) \leq C_2 ||u||_{\mathcal{T}_\sigma} ||v||_{\mathcal{T}_\sigma}, \ \forall u, v \in \mathcal{T}_\sigma$$

Assumptions (continued)

Induce $\mathcal{A}_\sigma : \mathcal{T}_\sigma \to \mathcal{T}_{-\sigma}$ so $\langle \mathcal{A}_\sigma u, v \rangle = a_\sigma(u, v)$, and choose $\mathcal{T}_*$ a subspace of $\mathcal{T}_\infty$:

- $\mathcal{T}_*$ is dense in $\mathcal{T}_\infty$ and in $\mathcal{T}_\sigma$ when $\mathcal{A}_\sigma u \in \mathcal{T}_0$ for all $u \in \mathcal{T}_*$

Assumptions (continued)

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- $\mathcal{T}_*$ is dense in $\mathcal{T}_\infty$ and in $\mathcal{T}_\sigma$ when $\mathcal{A}_\sigma u \in \mathcal{T}_0$ for all $u \in \mathcal{T}_*$
- $\lim_{\sigma\to\infty} ||\mathcal{A}_\sigma u - \mathcal{A}_\infty u||_{\mathcal{T}_{-\sigma}} = 0$ for all $u \in \mathcal{T}_*$

Subspace Parameterization

Let $\{W_{\sigma,h}\}_{h>0} \subset \mathcal{T}_\sigma$ be closed subspaces.

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- Reminiscent of Galerkin approximations, h meshing parameter
- Considering $\sigma \to \infty$ with h fixed
- Considering $\sigma \to \infty$, $h \to 0^+$ simultaneously
- Nonlocal vector-valued problems: σ is a horizon parameter

Assumptions (continued)

Let $h \in (0, h_0]$. Then we assume:

- For any $\sigma \in [0, \infty]$, $v \in \mathcal{T}_\sigma$, there exists sequence $v_n \in \{W_{\sigma,h_n}\}_{n=1}^{\infty}$ with $h_n \to 0$ and $||v - v_n||_{\mathcal{T}_\sigma} \to 0$ as $n \to \infty$

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- (**Asymptotic Density**) For any $v \in \mathcal{T}_\infty$, there exists sequence $v_n \in \{W_{\sigma,h_n}\}_{n=1}^{\infty}$ with $h_n \to 0$ and $||v - v_n||_{\mathcal{T}_\infty} \to 0$ as $n \to \infty$

Parameterized Variational Problems

Standard Problem: Given $f \in \mathcal{T}_0$, find $u_\sigma \in \mathcal{T}_\sigma$ such that

$$a_\sigma(u_\sigma, v) = (f, v)_{\mathcal{T}_0}, \ \forall v \in \mathcal{T}_\sigma$$

Subspace Problem: Find $u_{\sigma,h} \in W_{\sigma,h}$ such that

$$a_\sigma(u_{\sigma,h}, v) = (f, v)_{\mathcal{T}_0}, \ \forall v \in W_{\sigma,h}$$

Convergence Theorems

Theorem

As $\sigma \to \infty$,

$$||u_\sigma - u_\infty||_{\mathcal{T}_0} \to 0.$$

Theorem

Fix $\sigma \in [0, \infty]$. *Then there exists* $C > 0$ *(independent of h) so that*

$$||u_{\sigma,h} - u_\sigma||_{\mathcal{T}_\sigma} \leq C \inf_{v_{\sigma,h} \in W_{\sigma,h}} ||v_{\sigma,h} - u_\sigma||_{\mathcal{T}_\sigma} \to 0$$

as $h \to 0^+$.

Remark: This proof follows from coercivity and boundedness.

Convergence of Approximate Solutions

Assume the following additional assumptions hold:

$$W_{\infty,h} = \mathcal{T}_\infty \cap \left(\bigcap_{\sigma \geq 0} W_{\sigma,h} \right)$$

$$\lim_{\sigma \to \infty} a_\sigma(u_h, v_h) = a_\infty(u_h, v_h), \ \forall u_h, v_h \in W_{\infty,h}$$

(**Strong Continuity**) If $\{||w_{\sigma,h}||_{\mathcal{T}_\sigma}\}_{\sigma>0}$ is uniformly bounded and $w_{\sigma,h} \to 0$ in $\mathcal{T}_0$ as $\sigma \to \infty$, then

$$\lim_{\sigma \to \infty} a_\sigma(w_{\sigma,h}, v_h) = 0, \ \forall v_h \in W_{\infty,h}$$

Convergence of Approximate Solutions

Theorem

Suppose $h > 0$, previous assumptions hold. Then

$$||u_{\sigma,h} - u_{\infty,h}||_{\mathcal{T}_0} \to 0$$

as $\sigma \to \infty$.

Method of proof:

- Asymptotic compact embedding yields subsequence of $\{u_{\sigma_n,h}\}_{n=1}^{\infty}$ convergent to $u_{*,h}$ in $\mathcal{T}_0$
- Show decay of $a_\sigma(u_{\sigma,h} - u_{*,h}, v_h)$ and $a_\sigma(u_{*,h}, v_h) - a_\infty(u_{*,h}, v_h)$

Asymptotically Compatible Schemes

Definition

A family of convergent approximations $\{u_{\sigma,h}\}_{\sigma>0,h>0}$ is **asymptotically compatible** to u_∞ if whenever $\sigma_n \to \infty$, $h_n \to 0^+$, we obtain

$$||u_{\sigma_n,h_n} - u_\infty||_{\mathcal{T}_0} \to 0$$

Theorem

Under the aforementioned assumptions, $\{u_{\sigma,h}\}_{\sigma>0,h>0}$ is an asymptotically compatible family.

Nonlocal Model Setup

Interaction domain:

$$\Omega_{\mathcal{I}} = \{y \in \mathbb{R}^d \setminus \Omega, \text{dist}(y, \partial\Omega) \leq 1\}$$

Nonlocal operator:

$$\mathcal{L}u(x) = -2\int_{\Omega}(u(y) - u(x))\gamma(x,y)dy$$

The kernel $\gamma(\cdot,\cdot)$ is nonnegative, radial, and symmetric, with

$$\hat{\gamma}(|\xi|) = |\xi|^2\gamma(|\xi|) \in L^1_{\text{loc}}(\mathbb{R}^d)$$

Nonlocal Model Setup (cont'd)

Define rescaled kernels, $\delta \in (0,1]$:

$$\hat{\gamma}_\delta(|\xi|) := \frac{1}{\delta^d}\hat{\gamma}\left(\frac{|\xi|}{\delta}\right)$$

$$\gamma_\delta(|\xi|) := \frac{1}{\delta^{d+2}}\gamma\left(\frac{|\xi|}{\delta}\right)$$

Nonlocal Volume Constrained Problem:

$$\begin{cases} \mathcal{L}_0 u = f, \text{ on } \Omega \\ u = 0, \quad \text{ on } \partial\Omega \end{cases}$$

Nonlocal Model Setup (cont'd)

Energy space ($\delta \in (0, 1]$) define $\mathcal{S}_\delta$:

$$\{u \in L^2(\Omega_\delta) : \int_\Omega \int_{B_\delta(x)} \gamma_\delta(|x-y|)(u(x)-u(y))^2 dxdy < \infty, u|_{\Omega_{\mathcal{I}_\delta}} = 0\}$$

Inner product:

$$(u, v)_{\mathcal{S}_\delta} := \int_\Omega \int_{B_\delta(x)} \gamma_\delta(|x-y|)(u(x)-u(y))(v(x)-v(y))dxdy$$

Define $\mathcal{S}_0 := H_0^1(\Omega)$, with inner product/norm:

$$(u, v)_{\mathcal{S}_0} := \int_\Omega \triangledown u(x) \cdot \triangledown v(x) dx$$

$$||u||^2_{\mathcal{S}_0} := \int_\Omega |\triangledown u(x)|^2 dx$$

Nonlocal Model Setup (cont'd)

Bilinear form:

$$b_\delta(u,v) = \begin{cases} \int_\Omega \int_{B_\delta(x)} \gamma_\delta(|y-x|)(u(y)-u(x))(v(y)-v(x))dydx, \delta > 0 \\ \int_\Omega \bigtriangledown u(x) \cdot \bigtriangledown v(x) dx, \delta = 0 \end{cases}$$

Weak formulation: find $u_\delta \in \mathcal{S}_\delta$ such that for all $v \in \mathcal{S}_\delta$:

$$b_\delta(u_\delta, v) = (f, v)_{L^2}$$

Finite Element Spaces

Let $\delta \in (0,1)$, fix a triangulation τ_h of Ω_δ; define

$$V_{\delta,h} := \{v \in \mathcal{S}_\delta, v|_K \in P(K)\ \forall K \in \tau_h\}$$

Assumptions:

- Every function in $V_{\delta,h}$ vanishes outside Ω

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$$V_{\delta,h} := \{v \in \mathcal{S}_\delta,\, v|_K \in P(K)\ \forall K \in \tau_h\}$$

Assumptions:

- Every function in $V_{\delta,h}$ vanishes outside Ω
- For any $v \in \mathcal{S}_\delta$, there is a sequence $v_h \in V_{\delta,h}$ where $||v_h - v||_{\mathcal{S}_\delta} \to 0$

Setup of Asymptotic Framework

Galerkin Approximation:
Find $u_{\delta,h} \in V_{\delta,h}$ so $b_\delta(u_{\delta,h}, v) = (f, v)_{L^2(\Omega_\delta)}$ $\forall v \in V_{\delta,h}$

Family of subspaces for ND model:

$$\mathcal{T}_\sigma := \begin{cases} \mathcal{S}_{1/\sigma}, & \sigma \in [1, \infty] \\ \mathcal{S}_1, & \sigma \in (0, 1) \\ L^2_0(\Omega), & \sigma = 0 \end{cases}$$

Asymptotic Results

Lemma

For any $\alpha \in (0,2]$, *and a kernel* γ_δ *satisfying* $|\xi|^\alpha \gamma_\delta(|\xi|) \in L^1(\mathbb{R}^d)$, *there exists* $C = C(\Omega)$ *such that*

$$||u||^2_{\mathcal{S}_\delta} \leq C \left(\int_{\mathbb{R}^d} |\xi|^\alpha \gamma_\delta(|\xi|) d\xi \right)$$

for all $u \in H_0^{\alpha/2}(\Omega) \cap L_0^2(\Omega)$.

Lemma (Uniform Poincaré)

There exists $C > 0$ *so for all* $\delta \in (0,1]$,

$$||u||^2_{L^2(\Omega_\delta)} \leq C||u||^2_{\mathcal{S}_\delta}$$

Asymptotic Results (continued)

Lemma

For any $v \in C_C^\infty(\Omega)$ and $x \in \Omega$, there is a pointwise limit

$$\mathcal{L}_\delta v(x) \to -\triangle v(x)$$

and there exists a $C = C(d, v)$ such that

$$\sup_{0<\delta<1} \sup_{x\in\Omega} |\mathcal{L}_\delta(x)| \leq C$$

Lemma

For any $v \in C_C^\infty(\Omega)$, as $\delta \to 0^+$,

$$||\mathcal{L}_\delta v - (-\triangle v)||_{L^2(\Omega)} \to 0$$

Asymptotic Results (continued)

Definition

$\hat{V}_{\delta,h}$ denotes continuous piecewise linear FES with mesh τ_h (same as $V_{\delta,h}$)

Lemma (Asymptotic Density)

The family $\{\hat{V}_{\delta,h}\}_{h>0,\delta>0}$ *is asymptotically dense in* $\mathcal{S}_0$.

This subspace of H_0^1 chosen to approximate $\mathcal{S}_0$ as $h \to 0^+$

Asymptotic Results (continued)

Theorem (Asymptotic Convergence)

If $\hat{V}_{\delta,h} \subset V_{\delta,h}$ *then as* $\delta \to 0$ *and* $h \to 0$,

$$||u_{\delta,h} - u_0||_{L^2(\Omega)} \to 0$$

Use asymptotic framework with verified assumptions:

- $\mathcal{T}_\sigma$ replaced by $\mathcal{S}_{1/\sigma}$
- a_σ replaced by $b_{1/\sigma}$
- $\mathcal{A}_\sigma$ replaced by $\mathcal{L}_{1/\sigma}$
- $W_{\sigma,h}$ replaced by $V_{1/\sigma,h}$
- $\delta \leftrightarrow 1/\sigma$

Finite Element Subspace

Let τ_h be a triangulation, define

$$\mathcal{V}_h := \{v \in C(\overline{\Omega_\delta}), v|_K \in C^\infty(\overline{K}),\ K \in \tau_h, v|\Omega_{\mathcal{I}_\delta} = 0\}$$

Lemma (Convergence in Subspace)

Let $u, v \in \mathcal{V}_h$, then as $\delta \to 0$,

$$(\mathcal{L}_\delta u, v)_{L^2(\Omega_\delta)} - (\bigtriangledown u, \bigtriangledown v)_{L^2(\Omega_\delta)} \to 0$$

For any $u_h, v_h \in V_{0,h}$,

$$\lim_{\delta \to 0} b_\delta(u_h, v_h) = b_0(u_h, v_h)$$

Asymptotic Results (continued)

Lemma (Inverse Inequality)

If $V_{\delta,h} = V_{0,h} \subset \mathcal{S}_0$, there exists $C > 0$ independent of δ such that for all $u_h \in V_{\delta,h}$,

$$||u_h||_{\mathcal{S}_\delta} \leq C||u_h||_{L^2(\Omega)}$$

Theorem

Let $u_{\delta,h}$ and $u_{0,h}$ be discrete solutions to the variational problems. If $V_{\delta,h} = V_{0,h} \subset \mathcal{S}_0$, then

$$||u_{\delta,h} - u_{0,h}||_{\mathcal{S}_0} \to 0$$

as $h \to 0$.

Asymptotic Results (continued)

Theorem

Assume $V_{\delta,h} \subset \mathcal{S}_\delta$ is a finite element space containing all continuous piecewise linear functions, and that $V_{0,h} = \mathcal{S}_0 \cap \left(\bigcap_{\delta>0} V_{\delta,h}\right)$. Then

$$||u_{\delta,h} - u_{0,h}||_{L^2(\Omega)} \to 0$$

as $\delta \to 0^+$.

Asymptotic Results (continued)

Method of proof:

- Let $u_{*,h}$ be such that $||u_{\delta,h} - u_{*,h}||_{L^2(\Omega)} \to 0$

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Method of proof:

- Let $u_{*,h}$ be such that $||u_{\delta,h} - u_{*,h}||_{L^2(\Omega)} \to 0$
- Sufficient to show $b_0(u_{*,h}, v_h) = (f, v_h)$ by uniqueness

Asymptotic Results (continued)

Method of proof:

- Let $u_{*,h}$ be such that $||u_{\delta,h} - u_{*,h}||_{L^2(\Omega)} \to 0$
- Sufficient to show $b_0(u_{*,h}, v_h) = (f, v_h)$ by uniqueness
- Write $(f, v_h) - b_0(u_{*,h}, v_h) =$ $b_\delta(u_{\delta,h} - u_{*,h}, v_h) + [b_\delta(u_{*,h}, v_h) - b_0(u_{*,h}, v_h)]$

Asymptotic Results (continued)

Method of proof:

- Let $u_{*,h}$ be such that $||u_{\delta,h} - u_{*,h}||_{L^2(\Omega)} \to 0$
- Sufficient to show $b_0(u_{*,h}, v_h) = (f, v_h)$ by uniqueness
- Write $(f, v_h) - b_0(u_{*,h}, v_h) =$
  $b_\delta(u_{\delta,h} - u_{*,h}, v_h) + [b_\delta(u_{*,h}, v_h) - b_0(u_{*,h}, v_h)]$
- Show each term decays to 0

Asymptotic Results (continued)

Let $K \subset \tau_h$, define

$$\Gamma_K := \{x \notin K, \mathrm{dist}(x, K) \leq \delta\}$$

and write $b_\delta(w_{\delta,h}, v_h)$ as

$$\sum_{K \in \tau_h} \int_K \int_{K \cup \Gamma_K} \gamma_\delta(x' - x)(w_{\delta,h}(x') - w_{\delta,h}(x))(v_h(x') - v_h(x))dx'dx$$

Integrate over $K \times K$ and $K \times \Gamma_K$ and estimate from above by Cauchy-Schwarz.

Highlights of PD Model

- Bilinear form depends on bulk and shear modulus, same kernel as ND model
- Energy space: $u \in \mathcal{S}^*$ such that $u = 0$ on $\Omega_{\mathcal{I}_\delta}$ and $\{\int_\Omega \int_{B_\delta(x)} \gamma_\delta(|x' - x|)(\mathrm{Tr}(\mathcal{D}^* u)(x', x))^2 dx' dx < \infty\}$
- Utilize family of finite element subspaces parameterized by mesh size
- Asymptotic compatibility conserved for conforming finite element approximations
- Need continuous piecewise linear functions in the subspaces

Goals of Numerics

- Verify convergence results from analytic framework

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- Show how convergence rates depend on decay of δ and h

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- Verify convergence results from analytic framework
- Show how convergence rates depend on decay of δ and h
- Relationship of δ and h shown: $\delta = h$

Sample Nonlocal Problem Formulation

Define

$$\mathcal{L}_\delta u = 2\int_{-\delta}^{\delta} \gamma_\delta(s)(u(x+s)-u(x))ds$$

and consider

$$\begin{cases} -\mathcal{L}_\delta u = f, & x \in (0,1) \\ u = 0, & x \notin (0,1) \end{cases}$$

Spline Function

Define fourth order B-spline:

$$u_0(x) = \begin{cases} 0, & x < 0 \\ \frac{x^4}{120}, & 0 \leq x < 0.2 \\ -\frac{x^4}{30} + \frac{x^3}{30} - \frac{x^2}{100} + \frac{x}{750} - \frac{1}{15000}, & 0.2 \leq x < 0.4 \\ \frac{x^4}{20} - \frac{x^3}{10} + \frac{7x^2}{100} - \frac{x}{50} + \frac{31}{15000}, & 0.4 \leq x < 0.5 \end{cases}$$

extended symmetrically to $[0.5, 1]$. Notice u_0'' is continuous.

Collected Data

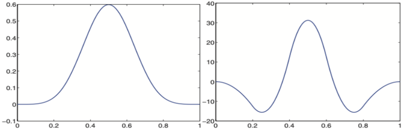

FIG. 2. *Graph of $u_0(x)$ and its second order derivative.*

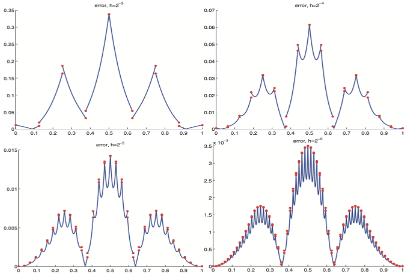

FIG. 3. *Pointwise error $u_{\delta,h}(x) - u_0(x)$ with $r = \frac{\delta}{h} = 3$ and $h = 2^{-k}$, $k = 3, 4, 5, 6$.*

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- Discretization errors more apparent for smaller h
- Modeling error less noticeable for smaller h (and δ)

- Q. Du, M. Gunzburger, R.B. Lehoucq, K. Zhou, *A nonlocal vector calculus, nonlocal volume-constrained problems, and nonlocal balance laws*, Math. Models Methods Appl. Sci., **23** (2013), pp. 493-540.
- X. Tian, Q. Du (2014). Asymptotically Compatible Schemes and Applications to Robust Discretization of Nonlocal Models. SIAM J. Numer. Anal. **52**(4), 1641-1665.

- Abner Salgado and the CAM organization committee
- Tadele Mengesha
- Xiaochuan Tian and Qiang Du

Q& A

Q: What other relationships are considered besides $\delta = h$?
A: $\delta = h^2$ or $\delta = \sqrt{h}$ (with some flexibility)
Q: Why wait so long to introduce $\lim_{\sigma \to \infty} a_\sigma(u_h, v_h) = a_\infty(u_h, v_h)$?
A: Only pertinent on subspaces when σ is not taken as fixed.
Q: Why assume $\lim_{\sigma \to \infty} ||\mathcal{A}_\sigma u - \mathcal{A}_\infty u||_{\mathcal{T}_{-\sigma}} = 0$ in $\mathcal{T}_*$?
A: Needed for convergence of solutions to variational problems