Integral Cauchy-Schwarz and Parseval's Identity

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Motivation

- Studied known proofs of Parseval's Identity

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- Searched for further convergence results

Outline

- Prerequisite Measure Theory

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- Proofs of Technical Lemmas for Integral Cauchy-Schwarz

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- Proofs of Technical Lemmas for Integral Cauchy-Schwarz
- Proof of Integral Cauchy-Schwarz Inequality
- Transition to Parseval's Identity
- Behavior in L^p

Main Results

Theorem (Integral Cauchy-Schwarz)

Let $E \subset \mathbb{R}^n$ be a bounded and measurable set, and let $g, h : E \to \mathbb{R}$ be bounded and measurable functions. Then

$$\left(\int_E g^2 d\mu\right)\left(\int_E h^2 d\mu\right) \geq \left(\int_E ghd\mu\right)^2. \tag{2.1}$$

Main Results (continued)

Theorem (Parseval's Identity on Positive Functions)

Let $D \subset \mathbb{R}^n$ be a bounded and measurable set, let $f : D \to \mathbb{R}$ be bounded, positive, and measurable on D, and let $\phi_n : D \to \mathbb{R}$ be a collection of functions which are mutually orthogonal on D with respect to $\frac{1}{f}$ for all $n \in \mathbb{N}^+$. Let the Fourier coefficients c_n be defined as

$$c_n := \frac{\int_D \phi_n d\mu}{\int_D \frac{\phi_n^2}{f} d\mu}. \tag{2.2}$$

and suppose the Fourier Expansion of $f = \sum_{n=1}^{\infty} c_n \phi_n$ exists. Then

$$\int_D f d\mu = \sum_{n=1}^{\infty} c_n^2 \int_D \frac{\phi_n^2}{f} d\mu. \tag{2.3}$$

Main Results (continued)

Theorem (Parseval's Identity)

Let $f : E \to \mathbb{R}$ be bounded and measurable, and let the sets D_i be bounded, measurable, and mutually disjoint such that $E = \cup_{i=1}^{\infty} D_i$. Assume that on each D_i, f carries a unique sign (is positive, negative, or zero) and has Fourier Coefficients denoted by

$$c_{i,n} := \frac{\int_{D_i} \phi_n d\mu}{\int_{D_i} \frac{\phi_n^2}{f} d\mu} \tag{2.4}$$

for each $i \in \mathbb{N}^+$, and suppose the Fourier Expansion of $f = \sum_{n=1}^{\infty} c_{i,n}\phi_n$ exists. Then

$$\int_E f d\mu = \sum_{i=1}^{\infty}\sum_{n=1}^{\infty} c_{i,n}^2 \int_{D_i} \frac{\phi_{i,n}^2}{f} d\mu. \tag{2.5}$$

Main Results (continued)

Theorem (L^p convergence)

Let $D \subset \mathbb{R}^n$ be a bounded, measurable subset of $\mathbb{R}^n$, and let $f : D \to \mathbb{R}$ be a measurable function for which $0 < f \leq 1$ on D. Fix $1 \leq p < \infty$. Choose $\{\phi_n\}_{n=1}^{\infty}$ mutually orthogonal on D w.r.t. the weight function $\frac{1}{f}$. Define the Fourier Coefficients and partial sums as before, and suppose the Fourier Expansion of $f = \sum_{n=1}^{\infty} c_n \phi_n$ exists. Then $s_N \to f$ in $L^p(D)$.

Our Measure Space

- $\mathcal{L}$: the Lebesgue measure on $\mathbb{R}^n$
- $\mathcal{M}$: the Lebesgue-measurable subsets of $\mathbb{R}^n$
- Our measure space: $(\mathbb{R}^n, \mathcal{M}, \mathcal{L})$
- Can also consider $(D, \mathcal{M} \cap \mathcal{P}(D), \mathcal{L})$ for a $D \subset \mathbb{R}^n$ measurable

Mutual Orthogonality

Definition (Mutual Orthogonality)

A family of [finite or countably many] functions $\{\phi_n\}$ is said to be **mutually orthogonal** with respect to a function g on a measurable set D if $\int_D \phi_m \phi_n g d\mu = 0$ whenever $m \neq n$.

We use positive weight functions of the form $\frac{1}{f}$.

L^p spaces

Definition (L^p space)

Let $1 \leq p < \infty$. A function f is in $L^p(D)$ for a measurable $D \subset \mathbb{R}^n$ if

$$||f||_{L^p(D)} := \left(\int_D |f|^p d\mu \right)^{\frac{1}{p}} < \infty, \tag{3.1}$$

in which case we say the L^p**-norm** of f is $||f||_{L^p(D)}$.

Definition (L^p convergence)

Let $1 \leq p < \infty$. A sequence of functions $f_n : D \to \mathbb{R}$ **converges in** L^p to a function $f : D \to \mathbb{R}$ if

$$||f_n - f||_{L^p(D)} \ \to \ 0. \tag{3.2}$$

as $n \to \infty$.

Integral Cauchy-Schwarz

Theorem (Integral Cauchy-Schwarz)

Let $E \subset \mathbb{R}^n$ be a bounded and measurable set, and let $g, h : E \to \mathbb{R}$ be bounded and measurable functions. Then

$$\left(\int_E g^2 d\mu\right)\left(\int_E h^2 d\mu\right) \geq \left(\int_E ghd\mu\right)^2. \tag{4.1}$$

Mean-Square Minimization Lemma

Lemma (Mean-Square Minimization)

Let $D \subset \mathbb{R}^n$ be a bounded and measurable set, and let $f, \phi_1 : D \to \mathbb{R}$ be bounded and measurable functions, where f only takes positive values in D. Then

$$\left(\int_D f d\mu\right)\left(\int_D \frac{\phi_1^2}{f} d\mu\right) \geq \left(\int_D \phi_1 d\mu\right)^2. \tag{4.2}$$

Construction of Orthogonal Family

- Fix $f, \phi_1 : D \to \mathbb{R}$
- Construct a mutually orthogonal family $\{\phi_n\}$ w.r.t. $\frac{1}{f}$ on D
- Can truncate after finitely many functions

In particular, $\forall i \neq j$,

$$\int_D \phi_i \phi_j \cdot \frac{1}{f} d\mu = 0 \tag{4.3}$$

Fourier Coefficients

Define the following:

$$c_n := \frac{\int_D \phi_n d\mu}{\int_D \frac{\phi_n^2}{f} d\mu} \tag{4.4}$$

$$s_N := \sum_{n=1}^{N} c_n \phi_n \tag{4.5}$$

If $\{\phi_n\}$ is finite we can truncate the sums and obtain an eventually constant sequence.

Mean-Square Deviation

The proof of the lemma is motivated by minimizing

$$\int_D (f - s_N)^2 \cdot \frac{1}{f} d\mu \qquad (4.6)$$

Expand and re-complete the square:

$$\int_D f d\mu - 2\sum_{n=1}^{N} c_n \int_D \phi_n d\mu + \sum_{n=1}^{N} c_n^2 \int_D \frac{\phi_n^2}{f} d\mu =$$

$$\sum_{n=1}^{N} \int_D \frac{\phi_n^2}{f} d\mu \left(c_n - \frac{\int_D \phi_n d\mu}{\int_D \frac{\phi_n^2}{f} d\mu} \right)^2 + \int_D f d\mu - \sum_{n=1}^{N} \frac{\left(\int_D \phi_n d\mu\right)^2}{\int_D \frac{\phi_n^2}{f} d\mu}. \qquad (4.7)$$

Mean-Square Deviation (continued)

Chose values of c_n to minimize this expression, so in fact

$$\int_D (f - s_N)^2 \cdot \frac{1}{f} d\mu = \int_D f d\mu - \sum_{n=1}^{N} \frac{\left(\int_D \phi_n d\mu\right)^2}{\int_D \frac{\phi_n^2}{f} d\mu}. \tag{4.8}$$

The left-hand side of (5.3) is nonnegative, so

$$\int_D f d\mu \geq \sum_{n=1}^{N} \frac{\left(\int_D \phi_n d\mu\right)^2}{\int_D \frac{\phi_n^2}{f} d\mu}. \tag{4.9}$$

Mean-Square Deviation (continued)

f positive, all terms of (4.9) nonnegative, so

$$\int_D f d\mu \;\geq\; \frac{\left(\int_D \phi_1 d\mu\right)^2}{\int_D \frac{\phi_1^2}{f} d\mu}. \tag{4.10}$$

Multiply across:

$$\left(\int_D f d\mu\right)\left(\int_D \frac{\phi_1^2}{f} d\mu\right) \;\geq\; \left(\int_D \phi_1 d\mu\right)^2. \tag{4.11}$$

Nonzero Cauchy-Schwarz

Lemma (Nonzero Cauchy-Schwarz)

Let $D \subset \mathbb{R}^n$ be a bounded and measurable set, and let $g, h : D \to \mathbb{R} \setminus \{0\}$ be bounded and measurable functions. Then

$$\left(\int_D g^2 d\mu\right)\left(\int_D h^2 d\mu\right) \geq \left(\int_D ghd\mu\right)^2. \tag{4.12}$$

Change of Variables

$$\left(\int_D f d\mu\right)\left(\int_D \frac{\phi_1^2}{f} d\mu\right) \geq \left(\int_D \phi_1 d\mu\right)^2. \tag{4.13}$$

Use the bijective change of variables $f = g^2$, $\phi_1 = gh$ on D. Obtain

$$\left(\int_D g^2 d\mu\right)\left(\int_D h^2 d\mu\right) \geq \left(\int_D gh d\mu\right)^2, \tag{4.14}$$

remarking f is positive on D.

Integral Cauchy-Schwarz

Theorem (Integral Cauchy-Schwarz)

Let $E \subset \mathbb{R}^n$ be a bounded and measurable set, and let $g, h : E \to \mathbb{R}$ be bounded and measurable functions. Then

$$\left(\int_E g^2 d\mu\right)\left(\int_E h^2 d\mu\right) \geq \left(\int_E ghd\mu\right)^2. \tag{4.15}$$

Filling in the Holes

Extend integral domains to zeros of g and h. Fix D:

$$D := \{x \in E, \ (g(x) \neq 0) \wedge (h(x) \neq 0)\} \tag{4.16}$$

$$E \setminus D = \{x \in E : (g(x) = 0) \vee (h(x) = 0)\} \tag{4.17}$$

Filling in the Holes (continued)

Notice $gh = 0$ on $E \setminus D$, so

$$\left(\int_D g^2 d\mu\right)\left(\int_D h^2 d\mu\right) \geq \left(\int_E ghd\mu\right)^2. \qquad (4.18)$$

Result follows after realizing $\int_E g^2 d\mu \geq \int_D g^2 d\mu$ and $\int_E h^2 d\mu \geq \int_D h^2 d\mu$. $\square$

Parseval's Identity on Positive Functions

Theorem (Parseval's Identity on Positive Functions)

Let $D \subset \mathbb{R}^n$ be a bounded and measurable set, let $f : D \to \mathbb{R}$ be bounded, positive, and measurable on D, and let $\phi_n : D \to \mathbb{R}$ be a collection of functions which are mutually orthogonal on D with respect to $\frac{1}{f}$ for all $n \in \mathbb{N}^+$. Let the Fourier coefficients c_n be defined as

$$c_n := \frac{\int_D \phi_n d\mu}{\int_D \frac{\phi_n^2}{f} d\mu}, \tag{5.1}$$

and suppose the Fourier Expansion of $f = \sum_{n=1}^{\infty} c_n \phi_n$ exists. Then

$$\int_D f d\mu = \sum_{n=1}^{\infty} c_n^2 \int_D \frac{\phi_n^2}{f} d\mu. \tag{5.2}$$

Outline of Proof

By taking the limit in

$$\int_D (f - s_N)^2 \cdot \frac{1}{f} d\mu = \int_D f d\mu - \sum_{n=1}^{N} \frac{\left(\int_D \phi_n d\mu\right)^2}{\int_D \frac{\phi_n^2}{f} d\mu}, \tag{5.3}$$

we know this inequality is an equality:

$$\int_D f d\mu \geq \sum_{n=1}^{N} \frac{\left(\int_D \phi_n d\mu\right)^2}{\int_D \frac{\phi_n^2}{f} d\mu}. \tag{5.4}$$

We know f's Fourier Expansion exists and is $f = \sum_{n=1}^{\infty} c_n \phi_n$. Take $N \to \infty$ in (5.4) and substitute the Fourier Coefficients to complete the proof. □

Generalizing to Non-signed Functions

Theorem (Parseval's Identity)

Let $f : E \to \mathbb{R}$ be bounded and measurable, and let the sets D_i be bounded, measurable, and mutually disjoint such that $E = \cup_{i=1}^{\infty} D_i$. Assume that on each D_i, f carries a unique sign (is positive, negative, or zero) and has Fourier Coefficients denoted by

$$c_{i,n} := \frac{\int_{D_i} \phi_n d\mu}{\int_{D_i} \frac{\phi_n^2}{f} d\mu} \tag{5.5}$$

for each $i \in \mathbb{N}^+$. Then

$$\int_E f d\mu = \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} c_{i,n}^2 \int_{D_i} \frac{\phi_{i,n}^2}{f} d\mu \tag{5.6}$$

Countable Additivity of Integration

Lemma (Countable Additivity of Integration)

Let f be a measurable function over the measurable set E. Let $\{E_m\}_{m=1}^{\infty}$ be a disjoint, countable collection of measurable subsets of E whose union is E. Then

$$\int_E f d\mu = \sum_{m=1}^{\infty} \int_{E_m} f d\mu. \tag{5.7}$$

Generalizing Proof to Non-signed Functions

Denote the following:

$$E_+ := \{x \in E, f(x) > 0\} \tag{5.8}$$

$$E_- := \{x \in E, f(x) < 0\} \tag{5.9}$$

$$E_0 := \{x \in E, f(x) = 0\} \tag{5.10}$$

These sets are measurable and disjoint, and $E = E_+ \cup E_- \cup E_0$.

Generalizing Proof to Non-signed Functions (continued)

By use of (5.7),

$$\int_E f d\mu = \int_{E_+} f d\mu + \int_{E_-} f d\mu \tag{5.11}$$

Applying (5.2) to f on E_+ and to $-f$ on E_- completes the proof.
□

The L^p space

Suppose $1 \leq p < \infty$.

Definition (L^p space)

A function f is in $L^p(D)$ for a measurable $D \subset \mathbb{R}^n$ if

$$||f||_{L^p(D)} := \left(\int_D |f|^p d\mu \right)^{\frac{1}{p}} < \infty, \tag{6.1}$$

in which case we say the L^p-**norm** of f is $||f||_{L^p(D)}$.

Convergence in L^p

Definition (L^p convergence)

We say that a sequence of functions $f_n : D \to \mathbb{R}$ **converges in** L^p to a function $f : D \to \mathbb{R}^n$ if

$$||f_n - f||_{L^p(D)} \ \to \ 0. \tag{6.2}$$

as $n \to \infty$. Equivalently,

$$\lim_{n\to\infty} \left(\int_D |f_n - f|^p d\mu \right)^{\frac{1}{p}} \ = \ 0 \tag{6.3}$$

Result: L^p convergence of Fourier Series

Theorem (L^p convergence)

Let $D \subset \mathbb{R}^n$ be a bounded, measurable subset of $\mathbb{R}^n$, and let $f : D \to \mathbb{R}$ be a bounded, measurable function. Fix $1 \leq p < \infty$. Choose $\{\phi_n\}_{n=1}^{\infty}$ mutually orthogonal on D w.r.t. the weight function $\frac{1}{f}$. Define the Fourier Coefficients and partial sums as before, and suppose the Fourier Expansion of $f = \sum_{n=1}^{\infty} c_n \phi_n$ exists. Then $s_N \to f$ in $L^p(D)$.

Proof Outline

$$\int_D (f - s_N)^2 \cdot \frac{1}{f} d\mu \tag{6.4}$$

- Special case: $f \leq 1$

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- General case, $f \leq M$

Proof Outline

$$\int_D (f - s_N)^2 \cdot \frac{1}{f} d\mu \tag{6.4}$$

- Special case: $f \leq 1$
- Use Integral Cauchy-Schwarz and Squeeze Theorem
- General case, $f \leq M$
- Scale Fourier Coefficients, linearity of Lebesgue integrals

$L^p(D)$ convergence lemma statement

Lemma

Let $D \subset \mathbb{R}^n$ be a bounded, measurable subset of $\mathbb{R}^n$, and let $f : D \to \mathbb{R}$ be a measurable function for which $0 < f \leq 1$ on D. Fix $1 \leq p < \infty$. Choose $\{\phi_n\}_{n=1}^{\infty}$ mutually orthogonal on D w.r.t. the weight function $\frac{1}{f}$. Define the Fourier Coefficients and partial sums as before, and suppose the Fourier Expansion of $f = \sum_{n=1}^{\infty} c_n \phi_n$ exists.

$L^p(D)$: Proof in $f \leq 1$ case

By Integral Cauchy-Schwarz,

$$\int_D |f - s_N|^2 \cdot \frac{1}{f} d\mu \int_D |f - s_N|^{2p-2} \cdot \frac{1}{f} d\mu \ \geq \ \left(\int_D |f - s_N|^p \cdot \frac{1}{f} d\mu \right)^2 . \tag{6.5}$$

LHS approaches 0 by proof of Parseval Identity; second LHS integral controlled due to boundedness of functions and set D

$L^p(D)$: Proof in $f \leq 1$ case (continued)

Since $f \leq 1$ on D,

$$\left(\int_D |f - s_N|^p \cdot \frac{1}{f} d\mu\right)^2 \geq \left(\int_D |f - s_N|^p d\mu\right)^2 \geq 0, \tag{6.6}$$

so $s_N \to f$ in $L^p(D)$ by Squeeze Theorem.

$L^p(D)$: Proof of General Case

- Suppose $f \leq M$ on D, let $g := \frac{f}{M}$.

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- Suppose $f \leq M$ on D, let $g := \frac{f}{M}$.
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$L^p(D)$: Proof of General Case

- Suppose $f \leq M$ on D, let $g := \frac{f}{M}$.
- Use special case on g
- The Fourier Coefficients of f are those of g scaled by factor of M

$L^p(D)$: Proof of General Case

- Suppose $f \leq M$ on D, let $g := \frac{f}{M}$.
- Use special case on g
- The Fourier Coefficients of f are those of g scaled by factor of M
- Linearity of Lebesgue Integrals yields general case

Next Steps

- Results on rate of convergence

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- Dissect other well-known integral identities

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Next Steps

- Results on rate of convergence
- Dissect other well-known integral identities
- Plancherel's Identity doesn't work
- Use methodology of proof to prove geometric and convolution-type inequalities

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