Recasting the Proof of Parseval's Equation

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Inequalities and Their Applications

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Outline

- Introduction, Motivations, Goals

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- Future Work

Motivation

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- New proof of Integral Cauchy-Schwarz Inequality
- Bridge between PDEs and measure theory

Main Results

Theorem (Integral Cauchy-Schwarz)

Let $E \subset \mathbb{R}$ be a bounded and measurable set, and let $g, h : E \to \mathbb{R}$ be bounded and measurable functions. Then

$$\left(\int_E g^2 d\mu\right)\left(\int_E h^2 d\mu\right) \geq \left(\int_E ghd\mu\right)^2. \tag{2.1}$$

Main Results (continued)

Theorem (Parseval's Equation on Positive Functions)

Let $D \subset \mathbb{R}$ be a bounded and measurable set, let $f : D \to \mathbb{R}$ be bounded, positive, and measurable on D, and let $\phi_n : D \to \mathbb{R}$ be a collection of functions which are mutually orthogonal on D with respect to $\frac{1}{f}$ for all $n \in \mathbb{N}^+$. Let the Fourier coefficients c_n be defined as

$$c_n := \frac{\int_D \phi_n d\mu}{\int_D \frac{\phi_n^2}{f} d\mu}, \tag{2.2}$$

Then

$$\int_D f d\mu = \sum_{n=1}^{\infty} c_n^2 \int_D \frac{\phi_n^2}{f} d\mu. \tag{2.3}$$

Main Results (continued)

Theorem (Parseval's Equation)

Let $f : E \to \mathbb{R}$ be bounded and measurable, and let the sets D_i be bounded, measurable, and mutually disjoint such that $E = \cup_{i=1}^{\infty} D_i$. Assume that on each D_i, f carries a unique sign (is positive, negative, or zero) and has Fourier Coefficients denoted by

$$c_{i,n} := \frac{\int_{D_i} \phi_n d\mu}{\int_{D_i} \frac{\phi_n^2}{f} d\mu} \tag{2.4}$$

for each $i \in \mathbb{N}^+$. Then

$$\int_E f d\mu = \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} c_{i,n}^2 \int_{D_i} \frac{\phi_{i,n}^2}{f} d\mu. \tag{2.5}$$

Measurability

Definition (Measurable Set in $\mathbb{R}$)

A set $E \subset \mathbb{R}$ is **measurable** with respect to an outer measure μ^* if for all $A \subset \mathbb{R}$,

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^C). \tag{3.1}$$

Definition (Measurable Function in $\mathbb{R}$)

Let $E \subset \mathbb{R}$ be a measurable set. A function $f : E \to \overline{\mathbb{R}}$ is **measurable** if

$$\{x \in \mathbb{R} : f(x) = c\} \tag{3.2}$$

is a measurable set $\forall\, c \in \overline{\mathbb{R}}$.

Our Measure Space

- $\mathcal{L}$: the Lebesgue measure on $\mathbb{R}$
- $\mathcal{M}$: the Lebesgue-measurable subsets of $\mathbb{R}$
- Our measure space: $(\mathbb{R}, \mathcal{M}, \mathcal{L})$

Countable Additivity of Integration

Lemma (Countable Additivity of Integration)

Let f be a measurable function over the measurable set E. Let $\{E_n\}_{n=1}^{\infty}$ be a disjoint, countable collection of measurable subsets of E whose union is E. Then

$$\int_E f d\mu = \sum_{n=1}^{\infty} \int_{E_n} f d\mu. \tag{3.3}$$

Other Definitions (continued)

Definition (Eventually Constant)

A sequence $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}^n$ is said to be **eventually constant** if $\exists m \in \mathbb{N}^+$ such that $x_m = x_k$ for all $k > m$.

Definition (Mutual Orthogonality)

A family of [finite or countably many] functions $\{\phi_n\}$ is said to be **mutually orthogonal** with respect to a function g on a measurable set D if $\int_D \phi_m \phi_n g d\mu = 0$ whenever $m \neq n$.

Integral Cauchy-Schwarz

Theorem (Integral Cauchy-Schwarz)

Let $E \subset \mathbb{R}$ be a bounded and measurable set, and let $g, h : E \to \mathbb{R}$ be bounded and measurable functions. Then

$$\left(\int_E g^2 d\mu\right)\left(\int_E h^2 d\mu\right) \geq \left(\int_E ghd\mu\right)^2. \tag{4.1}$$

Mean-Square Minimization Lemma

Lemma (Mean-Square Minimization)

Let $D \subset \mathbb{R}$ be a bounded and measurable set, and let $f, \phi_1 : D \to \mathbb{R}$ be bounded and measurable functions, where f only takes positive values in D. Then

$$\left(\int_D f d\mu\right)\left(\int_D \frac{\phi_1^2}{f} d\mu\right) \geq \left(\int_D \phi_1 d\mu\right)^2. \tag{4.2}$$

Construction of Orthogonal Family

- Fix $f, \phi_1 : D \to \mathbb{R}$
- Construct a mutually orthogonal family $\{\phi_n\}_{n=1}^{\infty}$ w.r.t. $\frac{1}{f}$
- Can truncate after finitely many functions

In particular, $\forall i \neq j$,

$$\int_D \phi_i \phi_j \cdot \frac{1}{f} d\mu = 0 \tag{4.3}$$

Fourier Coefficients

Define the following:

$$c_n := \frac{\int_D \phi_n d\mu}{\int_D \frac{\phi_n^2}{f} d\mu} \tag{4.4}$$

$$s_N := \sum_{n=1}^{N} c_n \phi_n \tag{4.5}$$

If $\{\phi_n\}$ is finite we can truncate the sums and obtain an eventually constant sequence.

Mean-Square Deviation

The proof of the lemma is motivated by minimizing

$$\int_D (f - s_N)^2 \cdot \frac{1}{f} d\mu \tag{4.6}$$

Mean-Square Deviation (continued)

Expand (4.6) and re-complete the square:

$$\int_D f d\mu - 2\sum_{n=1}^{N} c_n \int_D \phi_n d\mu + \sum_{n=1}^{N} c_n^2 \int_D \frac{\phi_n^2}{f} d\mu \; =$$

$$\sum_{n=1}^{N} \int_D \frac{\phi_n^2}{f} d\mu \left(c_n - \frac{\int_D \phi_n d\mu}{\int_D \frac{\phi_n^2}{f} d\mu} \right)^2 + \int_D f d\mu - \sum_{n=1}^{N} \frac{\left(\int_D \phi_n d\mu\right)^2}{\int_D \frac{\phi_n^2}{f} d\mu} . (4.7)$$

Mean-Square Deviation (continued)

Chose values of c_n to minimize this expression, so in fact

$$\min_{K \in \mathbb{N}^+} \int_D (f - s_K)^2 \cdot \frac{1}{f} d\mu = \int_D f d\mu - \sum_{n=1}^{N} \frac{\left(\int_D \phi_n d\mu\right)^2}{\int_D \frac{\phi_n^2}{f} d\mu}. \qquad (4.8)$$

The left-hand side of (4.8) is nonnegative, so

$$\int_D f d\mu \geq \sum_{n=1}^{N} \frac{\left(\int_D \phi_n d\mu\right)^2}{\int_D \frac{\phi_n^2}{f} d\mu}. \qquad (4.9)$$

Mean-Square Deviation (continued)

f positive, all terms of (4.9) nonnegative, so

$$\int_D f d\mu \;\geq\; \frac{\left(\int_D \phi_1 d\mu\right)^2}{\int_D \frac{\phi_1^2}{f} d\mu}. \tag{4.10}$$

Multiply across:

$$\left(\int_D f d\mu\right)\left(\int_D \frac{\phi_1^2}{f} d\mu\right) \;\geq\; \left(\int_D \phi_1 d\mu\right)^2. \tag{4.11}$$

Nonzero Cauchy-Schwarz

Lemma (Nonzero Cauchy-Schwarz)

Let $D \subset \mathbb{R}$ be a bounded and measurable set, and let $g, h : D \to \mathbb{R} \setminus \{0\}$ be bounded and measurable functions. Then

$$\left(\int_D g^2 d\mu\right)\left(\int_D h^2 d\mu\right) \geq \left(\int_D ghd\mu\right)^2. \tag{4.12}$$

Change of Variables

$$\left(\int_D f d\mu\right)\left(\int_D \frac{\phi_1^2}{f} d\mu\right) \geq \left(\int_D \phi_1 d\mu\right)^2. \tag{4.13}$$

Use the bijective change of variables $f = g^2$, $\phi_1 = gh$ on D. Obtain

$$\left(\int_D g^2 d\mu\right)\left(\int_D h^2 d\mu\right) \geq \left(\int_D gh d\mu\right)^2, \tag{4.14}$$

remarking f is positive on D.

Integral Cauchy-Schwarz

Theorem (Integral Cauchy-Schwarz)

Let $E \subset \mathbb{R}$ be a bounded and measurable set, and let $g, h : E \to \mathbb{R}$ be bounded and measurable functions. Then

$$\left(\int_E g^2 d\mu\right)\left(\int_E h^2 d\mu\right) \geq \left(\int_E gh d\mu\right)^2. \tag{4.15}$$

Filling in the Holes

Extend integrals to zeros of g and h. Fix D:

$$D := \{x \in E, (g(x) \neq 0) \wedge (h(x) \neq 0)\} \tag{4.16}$$

$$E \setminus D = \{x \in E : (g(x) = 0) \vee (h(x) = 0)\} \tag{4.17}$$

Filling in the Holes (continued)

Notice $gh = 0$ on $E \setminus D$, so

$$\left(\int_D g^2 d\mu\right)\left(\int_D h^2 d\mu\right) \geq \left(\int_E ghd\mu\right)^2. \tag{4.18}$$

Result follows after realizing $\int_E g^2 d\mu \geq \int_D g^2 d\mu$ and $\int_E h^2 d\mu \geq \int_D h^2 d\mu$. □

Parseval's Equation on Positive Functions

Theorem (Parseval's Equation on Positive Functions)

Let $D \subset \mathbb{R}$ be a bounded and measurable set, let $f : D \to \mathbb{R}$ be bounded, positive, and measurable on D, and let $\phi_n : D \to \mathbb{R}$ be a collection of functions which are mutually orthogonal on D with respect to $\frac{1}{f}$ for all $n \in \mathbb{N}^+$. Let the Fourier coefficients c_n be defined as

$$c_n := \frac{\int_D \phi_n d\mu}{\int_D \frac{\phi_n^2}{f} d\mu}, \tag{5.1}$$

Then

$$\sum_{n=1}^{\infty} c_n^2 \int_D \frac{\phi_n^2}{f} d\mu = \int_D f d\mu. \tag{5.2}$$

Outline of Proof

Recall we proved

$$\int_D f d\mu \geq \sum_{n=1}^{N} \frac{\left(\int_D \phi_n d\mu\right)^2}{\int_D \frac{\phi_n^2}{f} d\mu}. \tag{5.3}$$

and in fact this is equality by taking the limit as $N \to \infty$ in (4.8). Use that f's Fourier Expansion exists, so $f = \sum_{n=1}^{\infty} c_n \phi_n$. Take $N \to \infty$ in (5.3) and substitute (5.1) to complete the proof. □

Generalizing to Non-signed Functions

Theorem (Parseval's Equation)

Let $f : E \to \mathbb{R}$ be bounded and measurable, and let the sets D_i be bounded, measurable, and mutually disjoint such that $E = \cup_{i=1}^{\infty} D_i$. Assume that on each D_i, f carries a unique sign (is positive, negative, or zero) and has Fourier Coefficients denoted by

$$c_{i,n} := \frac{\int_{D_i} \phi_n d\mu}{\int_{D_i} \frac{\phi_n^2}{f} d\mu} \tag{5.4}$$

for each $i \in \mathbb{N}^+$. Then

$$\int_E f d\mu = \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} c_{i,n}^2 \int_{D_i} \frac{\phi_{i,n}^2}{f} d\mu \tag{5.5}$$

Generalizing Proof to Non-signed Functions

Denote the following:

$$E_+ := \{x \in E, f(x) > 0\} \tag{5.6}$$

$$E_- := \{x \in E, f(x) < 0\} \tag{5.7}$$

$$E_0 := \{x \in E, f(x) = 0\} \tag{5.8}$$

These sets are measurable and disjoint, and $E = E_+ \cup E_- \cup E_0$.

Generalizing Proof to Non-signed Functions (continued)

By use of (3.3),

$$\int_E f d\mu = \int_{E_+} f d\mu + \int_{E_-} f d\mu \tag{5.9}$$

Applying (5.2) to f on E_+ and to $-f$ on E_- completes the proof.
□

Next Steps

- Generalize to the Lebesgue Measure in $\mathbb{R}^n$

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- Generalize to other measures such as the spherical measure

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- Generalize to the Lebesgue Measure in $\mathbb{R}^n$
- Generalize to other measures such as the spherical measure
- Use methodology of proof to prove convolution inequalities

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