Nonlocal Optimal Control Problems with Quasiconvex Energy Densities in Varying Fractional Parameter

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Problem statement

Our family of nonlocal optimal control problems is to find

$$\mathcal{F}(\overline{u_{\delta,s}}, \overline{g_{\delta,s}}) = \min_{\substack{u \in H_0^{s,p,\delta}(\Omega_{-\delta};\mathbb{R}^n), \\ g \in Z_{\text{ad}}}} \left\{ \int_\Omega F(x, u(x))dx + \int_\Omega \Lambda(x)|g(x)|^{p'} dx \right\}$$

over pairs $(u, g) \in H_0^{s,p,\delta}(\Omega_{-\delta};\mathbb{R}^n) \times Z_{\text{ad}}$ that satisfy

$$u \in \text{argmin}_{v \in H_0^{s,p,\delta}(\Omega;\mathbb{R}^n)} \mathcal{W}_g^{\delta,s}(v)$$

Here $\mathcal{W}_g^{\delta,s}(\cdot)$ is either a nonlocal p-Laplacian energy or a general nonlocal quasiconvex energy!

Outline

Outline

Nonlocal models

- **Nonlocal models** are those that use integration to capture physical behaviors in lieu of differentiation
- Lets models assume far less regularity than PDE models
- Typical case study: peridynamics models for crack propagation

Prototype for nonlocal gradient (where $u : \Omega_\delta \to \mathbb{R}^n$)

$$\mathscr{G}_\rho u(x) \ := \ \int_\Omega \rho(x-y)\frac{u(x)-u(y)}{|x-y|} \otimes \frac{x-y}{|x-y|} dy,$$

where $\rho : \Omega \to \mathbb{R}$ is a kernel with a singularity at the origin.

The Fractional Riesz Gradients

Typical definition of Riesz s-fractional gradient:

$$\Omega := \mathbb{R}^n, \ \rho(x) := \frac{c_{n,s}}{|x|^{n+s-1}},$$

which produces a gradient operator

$$D^s u(x) := c_{n,s} \ P.V. \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+s}} \otimes \frac{x-y}{|x-y|} dy.$$

However, unbounded domains are not suitable for solid mechanics. A suitable alternative is this **truncated fractional gradient** for $\delta > 0$, $s \in (0,1)$

$$D^s_\delta u(x) := c_{n,s} \ P.V. \int_{B(x,\delta)} \frac{u(x)-u(y)}{|x-y|} \otimes \frac{x-y}{|x-y|} \frac{w_\delta(x-y)}{|x-y|^{n+s-1}} \, dy.$$

Polyconvexity

Definition (Polyconvex energy)

An energy $W : \Omega \times \mathbb{R}^n \times \mathbb{R}^{n^2} \to \mathbb{R}$ is said to be **polyconvex** if it is of the form

$$W(x, u, A) \; = \; W(x, u, A, \mathsf{cof}(A), \det(A))$$

and is jointly convex in A, $\mathsf{cof}(A)$, $\det(A)$.

Physical motivation:

- Dependence on $\mathsf{cof}(A)$ indicates explicit dependence on deformation of line elements
- Dependence on $\det(A)$ indicates explicit dependence on deformation of volume elements

NOTE: Weak continuity of determinant of gradient utilized to prove existence of minimizers!

Mooney-Rivlin materials

Example

The polyconvex family of **Compressible Mooney-Rivlin materials** are functions of the form

$$f(A) \ := \ a|A|^2 + b|\mathrm{cof}(A)|^2 + T(\det(A))$$

where $a, b > 0$ and $T : \mathbb{R} \to \mathbb{R}$ is the map defined by

$$T(d) \ := \ \begin{cases} \alpha d^2 - \beta \ln(d), \text{ if } d > 0 \\ +\infty, \text{ if } d \leq 0 \end{cases}$$

for some fixed $\alpha, \beta > 0$.

Quasiconvexity

Definition (Quasiconvex energy)

An energy $W : \Omega \times \mathbb{R}^n \times \mathbb{R}^{n^2} \to \mathbb{R}$ is said to be **quasiconvex** if for a.e. $x \in \Omega$, all $u \in \mathbb{R}^n$, and $A \in \mathbb{R}^{n^2}$ the following inequality holds for all $\psi \in W_0^{1,\infty}((0,1)^n; \mathbb{R}^n)$:

$$W(x, u, A) \ \leq \ \fint_{(0,1)^n} W(x, u, A + \nabla\psi(y))dy.$$

Physical motivation:

- An affine deformation has a smaller energy than the internally distorted deformation (distorted by ψ)

NOTE: In the classical context, Morrey's Theorem gives lower semi-continuity of energies, which in turn gives existence of minimizers

Novelties

The main contributions of this work are:

- Studying control problems where underlying function space has two nonlocal parameters (s fractional, δ horizon)
- Well-posedness of control problems with a convex energy constraint (minimizers are unique)
- Considering a control problem with an ill-posed state constraint (minimizers may not be unique)
- Relating nonlocal control problems to local ones posed on Sobolev Spaces ($s \to 1^-$)

Kernels and nonlocal gradient

Kernels

Horizon parameter: $\delta > 0$

- $w_\delta : \mathbb{R}^n \to [0, \infty)$ is radial; i.e. $w_\delta(x) = \bar{w}_\delta(|x|)$ for some non-negative $\bar{w}_\delta \in C_c^\infty([0, \infty))$, with $\text{supp}(\bar{w}_\delta) \subset [0, \delta)$.
- There is a constant $0 < b_0 < 1$ such that $\bar{w}|_{[0,b_0\delta]} = a_0$, where $a_0 = \max_{r \geq 0} \bar{w}_\delta(r)$.
- $\bar{w}_\delta(r_1) \geq \bar{w}_\delta(r_2)$ whenever $r_1 \leq r_2$.

Nonlocal gradient

- Consider $\Omega \subset \mathbb{R}^n$ bounded and open, $\Omega_{-\delta} := \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > \delta\}$, $\Omega_\delta := \Omega \cup \{x \in \mathbb{R}^n \mid \text{dist}(x, \partial\Omega) < \delta\}$
- Let $u \in C_0^\infty(\mathbb{R}^n; \mathbb{R}^n)$, and then the nonlocal gradient $D_\delta^s u$ is defined as

$$D_\delta^s u(x) \ := \ c_{n,s} \int_{B(x,\delta)} \frac{u(x) - u(y)}{|x-y|} \otimes \frac{x-y}{|x-y|} \frac{w_\delta(x-y)}{|x-y|^{n+s-1}} \ dy$$

Nonlocal function spaces

- Let $p \in [1, \infty)$. Then $H^{s,p,\delta}(\Omega; \mathbb{R}^n)$ is defined as the closure of $C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$ under the norm

$$\|u\|_{H^{s,p,\delta}(\Omega;\mathbb{R}^n)} := \|u\|_{L^p(\Omega_\delta;\mathbb{R}^n)} + \|D_\delta^s u\|_{L^p(\Omega;\mathbb{R}^{n^2})}.$$

- Zero nonlocal boundary data:

$$H_0^{s,p,\delta}(\Omega_{-\delta}; \mathbb{R}^n) := \{u \in H^{s,p,\delta}(\Omega; \mathbb{R}^n) \mid u = 0 \text{ on } \Omega \setminus \Omega_{-\delta}\}$$

- Arbitrary nonlocal boundary data (for $f \in H^{s,p,\delta}(\Omega; \mathbb{R}^n)$):

$$H_f^{s,p,\delta}(\Omega; \mathbb{R}^n) := f + H_0^{s,p,\delta}(\Omega_{-\delta}; \mathbb{R}^n)$$

The definition of nonlocal gradient is extended to functions in these spaces by density

Outline

Preliminaries

Proposition (Nonlocal Poincaré, Cueto/Kreisbeck/Schönberger 2023)

Let $s \in [0,1]$, $p \in (1,\infty)$, $\Omega \subset \mathbb{R}^n$ *be open and bounded. Then there exists a constant* $C > 0$ *depending only on* Ω, δ, *and* p *such that for all* $u \in H_0^{s,p,\delta}(\Omega;\mathbb{R}^n)$, *we have*

$$\|u\|_{L^p(\Omega;\mathbb{R}^n)} \leq C\|D_\delta^s u\|_{L^p(\Omega;\mathbb{R}^{n^2})}.$$

Proposition (Compact Embedding, Bellido/Cueto/Mora-Corral 2023)

Let $s \in (0,1)$, $p \in (1,\infty)$, $f \in H^{s,p,\delta}(\Omega;\mathbb{R}^n)$, *and suppose* $\{u_j\}_{j=1}^\infty \subset H_f^{s,p,\delta}(\Omega;\mathbb{R}^n)$ *is a sequence such that* $u_j \rightharpoonup u$ *weakly in* $H^{s,p,\delta}(\Omega;\mathbb{R}^n)$. *Then* $u_j \to u$ *strongly in* $L^q(\Omega;\mathbb{R}^n)$, *where* q *satisfies:*

- $q \in [1, p_s^*)$ *if* $sp < n$;
- $q \in [1,\infty)$ *if* $sp = n$;
- $q \in [1,\infty]$ *if* $sp > n$,

where $p_s^* := \frac{np}{n-sp}$. *In addition, we have that* $u \in H_f^{s,p,\delta}(\Omega;\mathbb{R}^n)$.

Convergences in varying fractional exponent

Proposition (Cueto/Kreisbeck/Schönberger 2023)

Let $p \in (1,\infty)$ with $\Omega \subset \mathbb{R}^n$ open and bounded. Suppose $\{s_j\}_{j=1}^{\infty} \subset [0,1]$ converges to 1 where $u_j \in H_0^{s_j,p,\delta}(\Omega;\mathbb{R}^n)$ is a sequence such that

$$\sup_{j\in\mathbb{N}^+} \|D_\delta^{s_j} u_j\|_{L^p(\Omega;\mathbb{R}^n)} < \infty.$$

Then there exists $u \in W_0^{1,p}(\Omega;\mathbb{R}^n)$ so that, up to a non-relabeled sub-sequence, $u_j \to u$ in $L^p(\Omega_\delta;\mathbb{R}^n)$ and $D_\delta^{s_j} u_j \rightharpoonup \nabla u$ weakly in $L^p(\Omega;\mathbb{R}^{n^2})$.

Proposition (Cueto/Kreisbeck/Schönberger 2023)

Let $p \in (1,\infty)$ and let $\{s_j\}_{j=1}^{\infty} \subset [0,1]$ be a sequence converging to 1. Suppose that $u \in W_0^{1,p}(\Omega;\mathbb{R}^n)$. Then $D_\delta^{s_j} u \to \nabla u$ strongly in $L^p(\Omega;\mathbb{R}^{n^2})$ as $j \to \infty$.

Outline

Nonlocal problem notation

- Let $\langle \cdot, \cdot \rangle$ denote the duality pairing between $L^p(\Omega;\mathbb{R}^n)$ and $L^{p'}(\Omega;\mathbb{R}^n)$, and $\cdot$ denote the Fröbenius inner product between matrices in $\mathbb{R}^{n^2}$
- Non-homogeneous nonlocal p-Laplacian form:

$$\mathcal{Y}_{\delta,s}(u,v) \;=\; \int_\Omega \mathfrak{a}(x)|D^s_\delta u(x)|^{p-2} D^s_\delta u(x) \cdot D^s_\delta v(x) dx,$$

  where $\mathfrak{a} \in L^\infty(\Omega)$ is a strictly positive coefficient.
- Weak form of constraint:

$$\begin{cases} \mathcal{Y}_{\delta,s}(u,v) \;=\; \langle g_{\delta,s}, v\rangle, & \forall v \in H_0^{s,p,\delta}(\Omega;\mathbb{R}^n) \\ u \;=\; 0, & x \in \Omega_\delta \setminus \Omega \end{cases}$$

- For this section only, the nonlocal energy density is

$$\mathcal{W}_g^{\delta,s}(u) \;:=\; \frac{1}{p}\mathcal{Y}_{\delta,s}(u,u) - \langle g, u\rangle$$

Local problem notation

- Local non-homogeneous p-Laplacian form:

$$\mathcal{Y}_0(u,v) \; = \; \int_\Omega \mathfrak{a}(x)|\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla v(x)dx$$

- Weak form of constraint:

$$\begin{cases} \mathcal{Y}_0(u,v) \; = \; \langle g,v\rangle & \forall v \in W_0^{1,p}(\Omega;\mathbb{R}^n) \\ u \; = \; 0, & x\in\partial\Omega \end{cases}$$

- For this section only, the local energy density is

$$\mathcal{W}_g^{\mathsf{loc}}(u) \; := \; \frac{1}{p}\mathcal{Y}_0(u,u) - \langle g,u\rangle$$

Admissible classes

- The admissible control space is Z_{ad}, a nonempty, closed, convex, and bounded subset of $L^{p'}(\Omega;\mathbb{R}^n)$, and it takes the form

$$Z_{\text{ad}} := \{z \in L^{p'}(\Omega;\mathbb{R}^n) \mid a(x) \preceq z(x) \preceq b(x) \text{ a.e. } x \in \Omega\}$$

  for some vector-valued functions $a, b \in L^{\infty}(\Omega;\mathbb{R}^n)$.
- Nonlocal admissible set:

$$\begin{aligned}\mathcal{A}_s^{\delta} := \{&(w_{\delta,s}, f_{\delta,s}) \in H_0^{s,p,\delta}(\Omega;\mathbb{R}^n) \times Z_{\text{ad}} \mid \\ &\mathcal{Y}_{\delta,s}(w_{\delta,s}, v_{\delta,s}) = \langle f_{\delta,s}, v_{\delta,s}\rangle,\ \forall v_{\delta,s} \in H_0^{s,p,\delta}(\Omega;\mathbb{R}^n)\}\end{aligned}$$

- Local admissible set:

$$\begin{aligned}\mathcal{A}^{\text{loc}} := \{&(w, f) \in W_0^{1,p}(\Omega;\mathbb{R}^n) \times Z_{\text{ad}} \mid \\ &\mathcal{Y}_0(w, v) = \langle f, v\rangle,\ \forall v \in W_0^{1,p}(\Omega;\mathbb{R}^n)\}\end{aligned}$$

Cost functional setup

The cost functional is of the form

$$\mathcal{F}(u,g) \;:=\; \int_\Omega F(x,u(x))dx + \int_\Omega \Lambda(x)|g(x)|^{p'}dx.$$

A special case is

$$\mathcal{F}(u,g) \;:=\; \frac{1}{p}\|u-u_{\mathsf{des}}\|^p_{L^p(\Omega;\mathbb{R}^n)} + \frac{\lambda}{p'}\|g\|^{p'}_{L^{p'}(\Omega;\mathbb{R}^n)},$$

where $\lambda > 0$ is a regularization parameter, and $u_{\mathsf{des}} \in L^p(\Omega;\mathbb{R}^n)$ is a fixed desired state.

Cost functional assumptions

We assume that $F : \Omega \times \mathbb{R}^n \to \mathbb{R}$ satisfies:

- For all $v \in \mathbb{R}^n$ the mapping $x \mapsto F(x, v)$ is measurable;
- For all $x \in \Omega$ the mapping $v \mapsto F(x, v)$ is continuous and convex;
- There exist constant $c_1 > 0$ and $l \in L^1(\Omega)$ for which

$$|F(x, v)| \leq \; c_1 |v|^{p_s^*} + l(x)$$

  for all $x \in \Omega$ and all $v \in \mathbb{R}^n$, where $p_s^* := \frac{np}{n-sp}$.

We also assume that $\Lambda \in L^\infty(\Omega)$ is strictly positive function, i.e. there exists $\gamma > 0$ such that $\Lambda(x) \geq \gamma$ for all $x \in \Omega$.

Well-posedness results

Proposition (Well-posedness of state equation)

For any fixed $g_{\delta,s} \in Z_{ad}$*, there is a unique weak solution* $u_{\delta,s} \in H_0^{s,p,\delta}(\Omega;\mathbb{R}^n)$ *to*

$$\mathcal{Y}_{\delta,s}(u_{\delta,s}, v_{\delta,s}) \ = \ \langle g_{\delta,s}, v_{\delta,s} \rangle, \quad \forall v_{\delta,s} \in H^{s,p,\delta}(\Omega;\mathbb{R}^n)$$

satisfying a stability estimate

$$\|u_{\delta,s}\|_{H^{s,p,\delta}(\Omega_\delta;\mathbb{R}^n)} \lesssim \ \|g_{\delta,s}\|_{L^{p'}(\Omega;\mathbb{R}^n)}$$

with constant independent of s*. In addition, the underlying solution mapping is compact.*

Theorem (Well-posedness of nonlocal control problem)

Let $\delta > 0$, $s \in (0,1)$ *be fixed. There exists a pair* $(\overline{u_{\delta,s}}, \overline{g_{\delta,s}}) \in \mathcal{A}_s^\delta$ *solving the nonlocal control problem. If* F *is strictly convex in the second argument, then this solution is unique.*

Γ-convergence of convex energies

This result holds for quasiconvex energies with p-growth bounds, but also for $\mathcal{W}_g^{\delta,s}(u) := \frac{1}{p}\mathcal{Y}_{\delta,s}(u,u) - \langle g, u\rangle$ and $\mathcal{W}_g^{\text{loc}}(u) := \frac{1}{p}\mathcal{Y}_0(u,u) - \langle g, u\rangle$ for a $g \in Z_{\text{ad}}$ arbitrary.

Theorem (Cueto/Kreisbeck/Schönberger 2023)

Let $\delta > 0$, $g \in Z_{ad}$ be fixed. Then the family of functionals $\{\mathcal{W}_g^{\delta,s}\}_{s<1}$ will Γ-converge in the strong $L^p(\Omega;\mathbb{R}^n)$ topology to $\mathcal{W}_g^{loc}$, which we will denote $\mathcal{W}_g^{\delta,s} \xrightarrow{\Gamma,\ s\to 1^-} \mathcal{W}_g^{loc}$. In other words, we have the following:

1. *If $\{u_s\}_{s<1} \subset L^p(\Omega;\mathbb{R}^n)$ is a sequence such that $u_s \to u$ strongly in $L^p(\Omega;\mathbb{R}^n)$, then we have the* ***lim-inf inequality***

$$\mathcal{W}_g^{loc}(u) \ \leq \ \liminf_{s\to 1^-} \mathcal{W}_g^{\delta,s}(u_s).$$

2. *If $u \in L^p(\Omega;\mathbb{R}^n)$, then there exists a* ***recovery sequence*** *of $\{u_s\}_{s<1} \subset L^p(\Omega;\mathbb{R}^n)$ such that $u_s \to u$ strongly in $L^p(\Omega;\mathbb{R}^n)$ and*

$$\mathcal{W}_g^{loc}(u) \ \geq \ \limsup_{s\to 1^-} \mathcal{W}_g^{\delta,s}(u_s).$$

Convergence of minimizers

Lemma (Compactness)

Let $\delta > 0$ be fixed, and let $\{(\overline{u_s}, \overline{g_s})\}_{s<1}$ denote the solutions to our family of convex nonlocal control problems. Then there exists a sequence $\{s_j\}_{j=1}^{\infty}$ such that $s_j \to 1$, and a pair $(\overline{u}, \overline{g}) \in \mathcal{A}^{loc}$ such that $\overline{u_s} \to \overline{u}$ strongly in $L^p(\Omega; \mathbb{R}^n)$ and $\overline{g_s} \rightharpoonup \overline{g}$ weakly in $L^{p'}(\Omega; \mathbb{R}^n)$, as $s \to 1^-$.

Theorem (Convergence of minimizers as $s \to 1^-$)

As $s \to 1^-$, we have that $\overline{u_s} \to \overline{u}$ strongly in $L^p(\Omega; \mathbb{R}^n)$ and $\overline{g_s} \to \overline{g}$ strongly in $L^{p'}(\Omega; \mathbb{R}^n)$. Moreover, we have the limit

$$\lim_{s \to 1^-} \mathcal{F}(\overline{u_s}, \overline{g_s}) = \mathcal{F}(\overline{u}, \overline{g}),$$

and $(\overline{u}, \overline{g})$ is the solution to the local control problem.

Strategy: compare the limit obtained by compactness to arbitrary test pairs $(u, g) \in \mathcal{A}^{\text{loc}}$, pick $\{u_s\}_{s<1}$ with $(u_s, g) \in \mathcal{A}_\delta^s$ as a recovery sequence

Stronger convergence of optimal states

Theorem

Let $\{(\overline{u_s}, \overline{g_s})\}_{s<1}$ *be the solutions to the nonlocal control problem, while* $(\overline{u}, \overline{g})$ *is the solution to the local control problem. Then we have that* $\overline{u_s} \to \overline{u}$ *converges strongly in* $H^{s,p,\delta}(\Omega;\mathbb{R}^n)$*, i.e.*

$$\lim_{s\to 1^-} \|\overline{u_s} - \overline{u}\|_{H^{s,p,\delta}(\Omega;\mathbb{R}^n)} = 0.$$

NOTE: This result rests on the structure of our functional moreso than its convexity!

Due to Γ-convergence:

$$\lim_{s\to 1^-} \mathcal{W}^{\delta,s}_{\overline{g_s}}(\overline{u_s}) = \mathcal{W}^{\text{loc}}_{\overline{g}}(\overline{u})$$

Stronger convergence of optimal states: proof

Now due to $\overline{u_s} \to \overline{u}$ in $L^p(\Omega;\mathbb{R}^n)$, $\overline{g_s} \to \overline{g}$ in $L^{p'}(\Omega;\mathbb{R}^n)$:

$$\lim_{s\to 1^-} \int_\Omega \mathfrak{a}(x)|D^s_\delta \overline{u_s}(x)|^p dx \;=\; \lim_{s\to 1^-} \int_\Omega \overline{u_s}(x)\overline{g_s}(x)\; dx \;=\; \int_\Omega \mathfrak{a}(x)|\nabla \overline{u}(x)|^p dx.$$

Since $D^s_\delta \overline{u_s} \rightharpoonup \nabla \overline{u}$ weakly in $L^p(\Omega;\mathbb{R}^{n^2})$ by compactness, this limit improves convergence to $D^s_\delta \overline{u_s} \to \nabla \overline{u}$ **strongly** in $L^p(\Omega;\mathbb{R}^{n^2})$. By Triangle Inequality:

$$\begin{aligned}
&\mathcal{Y}_{\delta,s}(\overline{u_s}-\overline{u},\overline{u_s}-\overline{u}) \;=\; \int_\Omega \mathfrak{a}(x)|D^s_\delta(\overline{u_s}-\overline{u})(x)|^p dx \;\le\\
&\int_\Omega \mathfrak{a}(x)(|D^s_\delta \overline{u_s}(x)-\nabla\overline{u}(x)|+|\nabla\overline{u}(x)-D^s_\delta\overline{u}(x)|)^p dx \;\lesssim\\
&\int_\Omega \mathfrak{a}(x)|D^s_\delta \overline{u_s}(x)-\nabla\overline{u}(x)|^p dx + \int_\Omega \mathfrak{a}(x)|\nabla\overline{u}(x)-D^s_\delta\overline{u}(x)|^p dx.
\end{aligned}$$

1st term goes to 0 due to convergence of optimal states, 2nd term goes to 0 due to strong convergence of gradient operators

Outline

Growth bounds

We assume that $W : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory energy density satisfying the bounds

$$c|A|^p - C \;\leq\; W(x,z,A) \;\leq\; C(1+|z|^p+|A|^p)$$

for a.e. $x \in \Omega$, $(z,A) \in \mathbb{R} \times \mathbb{R}^n$, and constants $c, C > 0$. Then we denote (for $u \in H^{s,p,\delta}(\Omega;\mathbb{R}^n)$)

$$\mathcal{W}_0^{\delta,s}(u) \;:=\; \int_\Omega W(x,u(x),D_\delta^s u(x))dx$$

and

$$\mathcal{W}_g^{\delta,s}(u) \;:=\; \int_\Omega W(x,u(x),D_\delta^s u(x))dx - \langle g,u\rangle$$

Local energy

We also denote (for $u \in W^{1,p}(\Omega;\mathbb{R}^n)$)

$$\mathcal{W}_0^{\mathsf{loc}}(u) \ := \ \int_\Omega W(x,u(x),\nabla u(x))dx$$

and

$$\mathcal{W}_g^{\mathsf{loc}}(u) \ := \ \int_\Omega W(x,u(x),\nabla u(x))dx - \langle g,u\rangle$$

Existence of minimizers for non-convex energy

Theorem (Cueto/Kreisbeck/Schönberger 2023)

Let $s \in (0,1)$, $\delta > 0$, $p \in (1,\infty)$, $\Omega \subset \mathbb{R}^n$ be open and bounded with $|\partial\Omega_{-\delta}| = 0$ and $g \in H^{s,p,\delta}(\Omega;\mathbb{R}^n)$. Suppose that $W : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory energy density satisfying the bounds

$$c|A|^p - C \;\leq\; W(x,z,A) \;\leq\; C(1+|z|^p+|A|^p)$$

for a.e. $x \in \Omega$, $(z,A) \in \mathbb{R}\times\mathbb{R}^n$. If $A \mapsto W(x,z,A)$ is quasiconvex a.e. $x \in \Omega_{-\delta}$ and all $z \in \mathbb{R}$, then the energy $\mathcal{W}_g^{s,\delta}$ defined by

$$\mathcal{W}_g^{\delta,s}(v) \;:=\; \int_\Omega W(x,v(x),D_\delta^s v(x))dx - \int_\Omega g(x)\cdot v(x)dx$$

admits a minimizer $u_{\delta,s} \in H_g^{s,p,\delta}(\Omega;\mathbb{R}^n)$.

NOTE: our results do not require W to be differentiable

Admissible sets

Nonlocal admissible set

$$\mathcal{Z}_s^\delta := \{(w_{\delta,s}, f_{\delta,s}) \in H_0^{s,p,\delta}(\Omega;\mathbb{R}^n) \times Z_{\mathsf{ad}} \mid \\ w_{\delta,s} \in \mathsf{argmin}_{v_{\delta,s}\in H_0^{s,p,\delta}(\Omega;\mathbb{R}^n)} \mathcal{W}^{\delta,s}_{f_{\delta,s}}[v_{\delta,s}]\}$$

Local admissible set

$$\mathcal{Z}^{\mathsf{loc}} := \{(w, f) \in W_0^{1,p}(\Omega;\mathbb{R}^n) \times Z_{\mathsf{ad}} \mid \\ w \in \mathsf{argmin}_{v\in W_0^{1,p}(\Omega;\mathbb{R}^n)} \mathcal{W}^{\mathsf{loc}}_f[v]\}$$

Lemmas for existence of minimizers

Lemma (Closure of Admissible Set)

Fix $\delta > 0$ and $s \in (0,1)$. Suppose that $\{(u_k, g_k)\}_{k=1}^{\infty} \subset \mathcal{Z}_s^{\delta}$ is a sequence such that $g_k \rightharpoonup \overline{g}$ weakly in $L^{p'}(\Omega;\mathbb{R}^n)$, and $u_k \rightharpoonup \overline{u}$ weakly in $H_0^{s,p,\delta}(\Omega;\mathbb{R}^n)$. Then the limiting pair $(\overline{u}, \overline{g}) \in \mathcal{Z}_s^{\delta}$.

Lemma (Boundedness of Admissible States)

Denote the set

$$U_{\delta,s} := \{u_{\delta,s} \in H_0^{s,p,\delta}(\Omega;\mathbb{R}^n) \mid \exists g_{\delta,s} \in Z_{ad},\ (u_{\delta,s}, g_{\delta,s}) \in \mathcal{Z}_s^{\delta}\}$$

as the collection of states which minimize the energy density for some admissible control. This set is bounded in the $H^{s,p,\delta}(\Omega;\mathbb{R}^n)$ norm.

NOTE: This is analogous to a stability estimate for well-posed minimization problems.

Existence of solutions for non-convex optimal control problem

Theorem

For any fixed $\delta > 0$ and $s \in [0, 1]$, the optimal control problem with the quasiconvex energy minimization constraint has a (not necessarily unique) solution $(\overline{u_{\delta,s}}, \overline{g_{\delta,s}}) \in \mathcal{Z}_s^\delta$.

- Functional bounded from below due to boundedness lemma
- Produce minimizing sequence $\{(u_k, g_k)\}_{k=1}^\infty \subset \mathcal{Z}_s^\delta$
- Produce limit $(\overline{u_{\delta,s}}, \overline{g_{\delta,s}})$ via reflexivity, belongs to $\mathcal{Z}_s^\delta$ due to closedness lemma
- Use lower semi-continuity to conclude

Γ-convergence of quasiconvex energies

Theorem (Cueto/Kreisbeck/Schönberger 2023)

Let $\delta > 0$, $g \in Z_{ad}$ be fixed. Then the family of functionals $\{\mathcal{W}_g^{\delta,s}\}_{s<1}$ will Γ-converge in the strong $L^p(\Omega;\mathbb{R}^n)$ topology to $\mathcal{W}_g^{loc}$, which we will denote $\mathcal{W}_g^{\delta,s} \xrightarrow{\Gamma,\ s\to 1^-} \mathcal{W}_g^{loc}$. In other words, we have the following:

1. *If $\{u_s\}_{s<1} \subset L^p(\Omega;\mathbb{R}^n)$ is a sequence such that $u_s \to u$ strongly in $L^p(\Omega;\mathbb{R}^n)$, then we have the **lim-inf inequality***

$$\mathcal{W}_g^{\delta,s}(u) \ \leq \ \text{liminf}_{s\to 1^-}\mathcal{W}_g^{loc}(u_s).$$

2. *If $u \in L^p(\Omega;\mathbb{R}^n)$, then there exists a **recovery sequence** of $\{u_s\}_{s<1} \subset L^p(\Omega;\mathbb{R}^n)$ such that $u_s \to u$ strongly in $L^p(\Omega;\mathbb{R}^n)$ and*

$$\mathcal{W}_g^{loc}(u) \ \geq \ \text{limsup}_{s\to 1^-}\mathcal{W}_g^{\delta,s}(u_s).$$

Convergence results as $s \to 1^-$

Let $\{(\overline{u_s}, \overline{g_s})\}_{s<1}$ be a family of solutions for the non-convex control problem.

Lemma (Lim-inf inequality)

Then there exists a pair $(\overline{u}, \overline{g}) \in \mathcal{Z}^{loc}$ *such that*

$$\mathcal{F}(\overline{u}, \overline{g}) \ \leq \ \mathit{liminf}_{s\to 1^-} \mathcal{F}(\overline{u_s}, \overline{g_s})$$

and $\overline{u_s} \to \overline{u}$ *strongly in* $L^p(\Omega; \mathbb{R}^n)$, $\overline{g_s} \rightharpoonup \overline{g}$ *weakly in* $L^{p'}(\Omega; \mathbb{R}^n)$

Lemma (Relative Minimization)

In addition, for every $g \in Z_{ad}$, *there exists a* $\widetilde{u} \in W_0^{1,p}(\Omega; \mathbb{R}^n)$ *so that* $(\widetilde{u}, g) \in \mathcal{Z}^{loc}$ *and*

$$\mathcal{F}(\overline{u}, \overline{g}) \ \leq \ \mathcal{F}(\widetilde{u}, g).$$

Let $g \in Z_{\text{ad}}$ be arbitrary and choose $u_s \in H_0^{s,p,\delta}(\Omega; \mathbb{R}^n)$ so that $(u_s, g) \in \mathcal{Z}_s^\delta$. Then produce a limit $\widetilde{u} \in W_0^{1,p}(\Omega; \mathbb{R}^n)$ through compactness, and $(\widetilde{u}, g) \in \mathcal{Z}^{\text{loc}}$

The issue is that we can't guarantee every local admissible state has a recovery sequence of nonlocal admissible states converging to it! This is where lack of uniqueness of minimizers for the energy is problematic!

Outline

Conjecture I

Conjecture (Asymptotic Compatibility)

Suppose $\{\delta_k\}_{k=0}^{\infty}$ *and* $\{s_k\}_{k=0}^{\infty}$ *are sequences of parameters such that* $\delta_k \to 0^+$ *and* $s_k \to 1^-$ *as* $k \to \infty$*. If* $\{(\overline{u_k}, \overline{g_k})\}_{k=1}^{\infty}$ *denotes the solution to our convex nonlocal control problem with parameters* δ_k *and* s_k*, and* $(\overline{u}, \overline{g})$ *denotes the solution to our local control problem, then:*

- $\overline{u_k} \to \overline{u}$ *strongly in* $L^p(\Omega; \mathbb{R}^n)$
- $\overline{g_k} \to \overline{g}$ *strongly in* $L^{p'}(\Omega; \mathbb{R}^n)$
- $\lim_{k\to\infty} \mathcal{F}(\overline{u_k}, \overline{g_k}) = \mathcal{F}(\overline{u}, \overline{g})$

Conjecture II

Conjecture (Compliance Functional)

If $\mathcal{W}_0^{\delta,s}(u) := \int_\Omega W(x, D_\delta^s u(x))dx$ *is a quasiconvex energy density for each* $\delta > 0$, $s \in (0,1)$, *let* $(\widehat{u_s}, \widehat{g_s}) \in \mathcal{Z}_s^\delta$ *denote a solution to the nonlocal control problem*

$$(\widehat{u_s}, \widehat{g_s}) \in \textit{argmin}_{(u_s, g_s)\in \mathcal{Z}_s^\delta} \int_\Omega g_s(x) \cdot u_s(x) dx.$$

Then there exists a sub-sequence of indices $\{s_k\}_{k=1}^\infty$ *with* $s_k \to 1^-$ *and a pair* $(\widehat{u}, \widehat{g}) \in \mathcal{Z}^{loc}$ *that solves the local control problem*

$$(\widehat{u}, \widehat{g}) \in \textit{argmin}_{(u,g)\in \mathcal{Z}^{loc}} \int_\Omega g(x) \cdot u(x) dx,$$

and we have the limit

$$\lim_{k\to\infty} \int_\Omega \widehat{g_{s_k}}(x) \cdot \widehat{u_{s_k}}(x) dx = \int_\Omega \widehat{g}(x) \cdot \widehat{u}(x) dx.$$

Summary

Accomplishments:

- Well-posedness of a nonlocal control problem with a convex constraint
- Existence of minimizers of a nonlocal control problem with a quasi/polyconvex constraint
- Convergence of minimizers for the nonlocal control problems with a convex constraint
- Weaker convergence properties in the limit for the nonlocal control problems with a quasi/polyconvex constraint