Analysis and Discretization of Optimal Control Problems in Peridynamics

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Problem statement

Find $(\overline{u_\delta}, \overline{g_\delta}) \in X_0 \times Z_{\text{ad}}$ such that

$$I(\overline{u_\delta}, \overline{g_\delta}) = \min_{g_\delta \in L^2, u_\delta \in X_0} \left\{ \int_\Omega F(x, u_\delta(x))dx + \frac{\lambda}{2} \int_\Omega \Gamma(x)|g_\delta(x)|^2 dx \right\},$$

subject to the strongly coupled system of equations

$$\mathcal{L}_\delta u_\delta = g_\delta$$

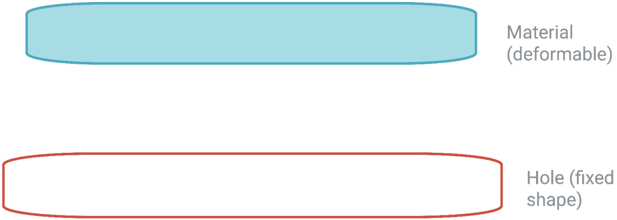
- $\delta \geq 0$ is the degree of non-locality (the **horizon parameter** in peridynamics);
- $\overline{g_\delta}$ is a [vector-valued] external force;
- $\overline{u_\delta}$ represents the [vector-valued] displacement.
- $\mathcal{L}_\delta$ is a non-local operator, exact formula to be given.

Outline

Sample candidate integrand

Suppose we want to deform our material to fit a predetermined hole of shape u_{des} as closely as possible. Then our candidate integrand is

$$F(x, u(x)) = |u(x) - u_{\text{des}}(x)|^2$$



Goals

- Show existence and uniqueness of minimizers
- Consider behavior as $\delta \to 0^+$
- Discretize via FEMs
- Study simultaneous limit as $\delta, h \to 0^+$ (asymptotic compatibility)

Bi-linear forms

Bi-linear form induces non-local state equation:

$$B_\delta(u,v) \;:=\; \iint_{\mathcal{D}_\delta} \mathfrak{A}(x,y) k_\delta(x-y) \frac{Du(x,y)}{|x-y|} \frac{Dv(x,y)}{|x-y|} dxdy \;=\; \int_\Omega g(x)\cdot v(x) dx$$

- $\mathfrak{A}$ is positive, bounded, symmetric
- k_δ a kernel with suitable properties
- $\mathcal{D}_\delta := (\Omega \times \Omega_\delta) \cup (\Omega_\delta \times \Omega)$, where $\Omega_\delta := \Omega \cup \{x \in \mathbb{R}^n, \text{dist}(x, \partial\Omega) < \delta\}$
- Du is projected difference, $Du(x,y) := (u(x) - u(y)) \cdot \frac{x-y}{|x-y|}$
- Models arise in peridynamics, specifically the bond-based model

Function spaces

Our function space is based on B_δ :

$$X(\Omega_\delta;\mathbb{R}^n) := \{u|_\Omega \in L^2(\Omega;\mathbb{R}^n), B_\delta(u,u) < \infty\}$$

Version with zero non-local boundary data:

$$X_0(\Omega_\delta;\mathbb{R}^n) := \{u \in X(\Omega_\delta;\mathbb{R}^n), u = 0 \text{ in } \Omega_\delta \setminus \Omega\}$$

State equation is well-posed!

Proposition (Nonlocal Poincaré-Korn (Mengesha-Du 2014))

There exists a $\delta_0 > 0$ *and a constant* $C(\delta_0) > 0$ *such that for all* $\delta \in (0, \delta_0]$ *and* $u \in X_0(\Omega_\delta; \mathbb{R}^n)$,

$$\|u\|^2_{L^2(\Omega;\mathbb{R}^n)} \leq \; C(\delta_0) \int_{\Omega_\delta} \int_{\Omega_\delta} k_\delta(x-y) \frac{|Du(x,y)|^2}{|x-y|^2} dxdy.$$

NOTE: The projected difference vanishes under infinitesimal rigid displacements.

Theorem (Existence and Uniqueness for State Equation)

For any $g_\delta \in L^2$, *there exists a unique* $u_\delta \in X_0$ *such that the state system*

$$B_\delta(u_\delta, w_\delta) \; = \; \langle g_\delta, w_\delta \rangle$$

is satisfied for all $w_\delta \in X_0$. *Furthermore, we have the stability estimate*

$$\|u_\delta\|_{X(\Omega_\delta;\mathbb{R}^n)} \lesssim \; \|g_\delta\|_{L^2(\Omega_\delta;\mathbb{R}^n)}$$

for some constant independent of δ.

Cost functional assumptions

$$I(u,g) \;:=\; \int_\Omega F(x,u(x))dx + \frac{\lambda}{2}\int_\Omega \Gamma(x)|g(x)|^2 dx$$

Here Z_{ad} is a box constraint, and $F : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is such that:

1. For all $v \in \mathbb{R}^n$ the mapping $x \mapsto F(x,v)$ is measurable;
2. For all $x \in \Omega$ the mapping $v \mapsto F(x,v)$ is continuous and convex;
3. There exist $c_1 > 0$ and $\ell \in L^1(\Omega)$
$$|F(x,v)| \leq \; c_1|v|^2 + \ell(x)$$

for all $x \in \Omega$, $v \in \mathbb{R}^n$. Moreover, $\Gamma \in L^\infty(\Omega)$ is positive a.e. and $\lambda \geq 0$

Well-posedness of optimal control problem

Theorem (Well-posedness)

For any $\delta > 0$: there exists $(\overline{u_\delta}, \overline{g_\delta}) \in X_0(\Omega_\delta; \mathbb{R}^n) \times Z_{ad}$ minimizing

$$I(u_\delta, g_\delta) = \int_\Omega F(x, u_\delta(x))dx + \frac{\lambda}{2} \int_\Omega \Gamma(x)|g_\delta(x)|^2 dx,$$

where $\overline{u_\delta} \in X_0$ solves

$$B_\delta(u_\delta, v_\delta) = \int_\Omega g_\delta(x) \cdot v_\delta(x) dx \quad \forall v_\delta \in X_0$$

Furthermore, if F is strictly convex or $\lambda > 0$, then the minimizer is unique.

Optimality conditions

Let S_δ denote solution operator for state equation
Non-local continuous optimality conditions (including adjoint):

$$
\begin{aligned}
\overline{u_\delta} &= S_\delta \overline{g_\delta} \\
\overline{p_\delta} = S_\delta^* F_u(\cdot, \overline{u_\delta}) &= S_\delta F_u(\cdot, \overline{u_\delta}) \\
\langle \overline{p_\delta} + \lambda \overline{g_\delta}, \gamma_z - \overline{g_\delta} \rangle_{L^2} &\geq 0, \quad \forall \gamma_z \in Z_{\mathrm{ad}}.
\end{aligned}
$$

Projection formula:

$$
\overline{g_\delta}(x) = -\frac{1}{\lambda} \mathbb{P}_{Z_{\mathrm{ad}}}(\overline{p_\delta}(x))
$$

NOTE: No second-order optimality conditions needed (strict convexity!)

Convergence of state equation as $\delta \to 0^+$

Local bi-linear form (of Navier-Lamé system of linear elasticity)

$$B_0(u,v) \ := \ C(n) \int_\Omega \mathfrak{a}(x)(2\langle \mathrm{Sym}(\nabla u), \mathrm{Sym}(\nabla v)\rangle_F + \mathrm{div}(u)\mathrm{div}(v))dx$$

where $C(n) := \frac{1}{(n+2)(n+4)}$.

Theorem

Suppose $\{(\overline{u_\delta}, \overline{g_\delta})\}_{\delta>0}$ is the family of solutions to the non-local problem. Then, there is $(\overline{u}, \overline{g})$ such that $\overline{u_\delta} \to \overline{u}$ in $L^2(\Omega;\mathbb{R}^n)$ and $\overline{g_\delta} \rightharpoonup \overline{g}$ in $L^2(\Omega;\mathbb{R}^n)$. Moreover, $(\overline{u}, \overline{g})$ solves the local optimal control problem.

Local state equation that $(\overline{u}, \overline{g})$ satisfies the constraint:

$$B_0(u,v) \ = \ \int_\Omega g(x)\cdot v(x)dx$$

Can improve convergence of controls to strong L^2 using projection formula

Discrete Formulation

- Use continuous piecewise linears for states
- Use piecewise constants for controls
- Discrete optimal control problem is well-posed

Optimality conditions (discrete)

Let $S_{\delta,h}$ denote solution operator for state equation

Non-local discrete optimality conditions (including adjoint):

$$\begin{aligned} \overline{u_{\delta,h}} &= S_{\delta,h}\overline{g_{\delta,h}} \\ \overline{p_{\delta,h}} = S^*_{\delta,h}F_u(\cdot,\overline{u_{\delta,h}}) &= S_{\delta,h}F_u(\cdot,\overline{u_{\delta,h}}) \\ \langle \overline{p_{\delta,h}} + \lambda\overline{g_{\delta,h}}, \gamma_z - \overline{g_{\delta,h}}\rangle_{L^2} &\geq 0, \quad \forall \gamma_z \in Z_{\text{ad}} \cap Z_h. \end{aligned}$$

Projection formula:

$$\overline{g_{\delta,h}}(x) = -\frac{1}{\lambda}\mathbb{P}_{Z_{\text{ad}}}(\Pi_0\overline{p_{\delta,h}}(x))$$

Here Π_0 is defined as $\Pi_0 z := \fint_T z$ on each triangle T in our mesh, and Z_h is the piecewise constants

Intermediary Functions

By Lax-Milgram, we may define $\widehat{u_\delta}, \widehat{p_\delta} \in X_0(\Omega_\delta; \mathbb{R}^n)$ such that

$$B_\delta(\widehat{u_\delta}, v_\delta) = \langle \overline{g_{\delta,h}}, v_\delta \rangle \quad \forall v_\delta \in X_0(\Omega_\delta; \mathbb{R}^n);$$

$$B_\delta(v_\delta, \widehat{p_\delta}) = \langle v_\delta, \overline{u_{\delta,h}} \rangle \quad \forall v_\delta \in X_0(\Omega_\delta; \mathbb{R}^n);$$

also define $\widehat{u_h}, \widehat{p_h} \in H_0^1(\Omega; \mathbb{R}^n)$ such that

$$B_0(\widehat{u_h}, v) = \langle \overline{g_h}, v \rangle \quad \forall v \in H_0^1(\Omega; \mathbb{R}^n);$$

$$B_0(v, \widehat{p_h}) = \langle v, \widehat{u_h} \rangle \quad \forall v \in H_0^1(\Omega; \mathbb{R}^n).$$

Non-local problem convergence: summary

Theorem (Convergence)

In the setting of our problems,

$$\|\overline{u_\delta} - \overline{u_{\delta,h}}\|_{X(\Omega_\delta;\mathbb{R}^n)} \lesssim \omega(h) + \inf_{v_{\delta,h}\in X_{\delta,h}} \|\widehat{u_\delta} - v_{\delta,h}\|_{X(\Omega_\delta;\mathbb{R}^n)} + \inf_{v_{\delta,h}\in X_{\delta,h}} [\overline{u_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)} + \inf_{v_{\delta,h}\in X_{\delta,h}} [\overline{p_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)};$$

$$\|\overline{g_\delta} - \overline{g_{\delta,h}}\|^2_{L^2(\Omega;\mathbb{R}^n)} \lesssim \omega(h)^2 + \left(\inf_{v_{\delta,h}\in X_{\delta,h}} [\overline{u_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)}\right)^2 + \left(\inf_{v_{\delta,h}\in X_{\delta,h}} [\overline{p_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)}\right)^2.$$

$$\|\overline{p_\delta} - \overline{p_{\delta,h}}\|_{X(\Omega_\delta;\mathbb{R}^n)} \lesssim \omega(h) + \inf_{v_{\delta,h}\in X_{\delta,h}} \|\widehat{p_h} - v_{\delta,h}\|_{X(\Omega_\delta;\mathbb{R}^n)} + \inf_{v_{\delta,h}\in X_{\delta,h}} \|\widehat{u_\delta} - v_{\delta,h}\|_{X(\Omega_\delta;\mathbb{R}^n)} + \inf_{v_{\delta,h}\in X_{\delta,h}} [\overline{u_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)} + \inf_{v_{\delta,h}\in X_{\delta,h}} [\overline{p_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)}.$$

Non-local problem convergence: comments

- Idea for control proof attributed to D'Elia, Glusa, Otárola
- Here $\omega(h)$ represents the approximation properties of the projection Π_0
- $X_{\delta,h}$ is our discrete non-local space
- If our kernel is of fractional type, i.e. $k_\delta(\xi) \sim \frac{1}{|\xi|^{n+2s}} \chi_{B(0,\delta)}(\xi)$, then $\omega(h) \sim h^s$

Discrete analogue of convergence

Theorem (Discrete Convergence)

Suppose $\{(\overline{u_{\delta,h}}, \overline{g_{\delta,h}})\}_{\delta>0}$ is the family of solutions to the non-local discrete problem. Then, there is $(\overline{u_h}, \overline{g_h})$ such that $\overline{u_{\delta,h}} \to \overline{u_h}$ in $L^2(\Omega;\mathbb{R}^n)$ and $\overline{g_{\delta,h}} \to \overline{g_\delta}$ in $L^2(\Omega;\mathbb{R}^n)$. Moreover, $(\overline{u_h}, \overline{g_h})$ solves the local discrete optimal control problem.

What is asymptotic compatibility?

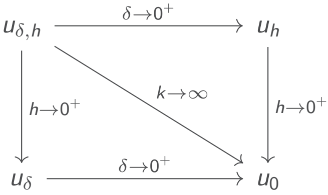
- Introduced by X. Tian and Q. Du (2014)
- Originally developed for linear, non-local systems

$$\mathcal{L}_{\delta,h} u_{\delta,h} = f$$

- Unconditional convergence of approximations in both discretization and horizon parameters

Definition (Asymptotic Compatibility)

Given fixed data f in a Hilbert Space, the family of solutions $\{u_{\delta,h}\}_{\delta,h>0}$ is **asymptotically compatible** in $\delta, h > 0$ if for any sequences $\{\delta_k\}_{k=1}^{\infty}, \{h_k\}_{k=1}^{\infty}$ with $\delta_k, h_k \to 0$, we have that $u_{\delta_k,h_k} \to u_0$ strongly in some Hilbert space norm, where u_0 is the solution to a local, continuous problem.

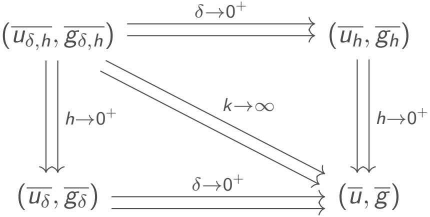


What is asymptotic compatibility? (continued)

Definition (Asymptotic Compatibility for Optimal Control)

We say that the family of solutions $\{(\overline{u_{\delta,h}}, \overline{g_{\delta,h}})\}_{h>0,\delta>0}$ to the nonlocal discrete optimal control problem is **asymptotically compatible** in $\delta, h > 0$ if for any sequences $\{\delta_k\}_{k=1}^{\infty}, \{h_k\}_{k=1}^{\infty}$ with $\delta_k, h_k \to 0$, we have that $\overline{g_{\delta_k,h_k}} \to \overline{g}$ strongly in $L^2(\Omega;\mathbb{R}^n)$, and $\overline{u_{\delta_k,h_k}} \to \overline{u}$ strongly in $L^2(\Omega;\mathbb{R}^n)$. Here $(\overline{u}, \overline{g}) \in H_0^1(\Omega;\mathbb{R}^n) \times Z_{\text{ad}}$ denotes the optimal solution for the local continuous problem.

NOTE: In particular, $H_0^1(\Omega;\mathbb{R}^n)$ is the limiting space of $X_0(\Omega_\delta;\mathbb{R}^n)$ as $\delta \to 0^+$



Asymptotic compatibility: result and notation

Theorem

Assume $\mathfrak{A}$ is Lipschitz. Then, our family of optimal control problems is asymptotically compatible as $\delta, h \to 0^+$.

Define $A_\delta : X_0(\Omega_\delta; \mathbb{R}^n) \to (X_0(\Omega_\delta; \mathbb{R}^n))^*$ as the invertible operator such that

$$\langle A_\delta u, v \rangle_{X_0^*, X_0} \ = \ B_\delta(u, v) \quad \forall u, v \in X_0(\Omega_\delta; \mathbb{R}^n);$$

define $A_0 : H_0^1(\Omega; \mathbb{R}^n) \to H^{-1}(\Omega; \mathbb{R}^n)$ as the invertible operator such that

$$\langle A_0 u, v \rangle_{H^{-1}, H_0^1} \ = \ B_0(u, v) \quad \forall u, v \in H_0^1(\Omega; \mathbb{R}^n).$$

Proof of asymptotic compatibility

Needed preliminaries:

1. Given a $v \in H_0^1(\Omega; \mathbb{R}^n)$, and $h_k, \delta_k \to 0$, we can find a sequence $v_k \in X_{\delta_k, h_k}$ such that $v_k \to v$ strongly in $H^1(\Omega; \mathbb{R}^n)$ as $k \to \infty$.
2. For any sequences $\{\delta_k\}_{k=1}^{\infty}, \{h_k\}_{k=1}^{\infty}$ with $\delta_k, h_k \to 0$, there exists a $C > 0$ so that $\|\overline{u_{\delta_k, h_k}}\|_{X(\Omega_{\delta_k}; \mathbb{R}^n)} \leq C$ uniformly in $k \in \mathbb{N}^+$.
3. We have that $A_\delta u \in L^2(\Omega; \mathbb{R}^n)$ and $\lim_{\delta \to 0^+} \|A_\delta u - A_0 u\|_{L^2(\Omega; \mathbb{R}^n)} = 0$ for each $u \in C_0^\infty(\Omega; \mathbb{R}^n)$.

Pick sub-sequence of $\{(\overline{u_k}, \overline{g_k}, \overline{p_k})\}_{k=1}^{\infty}$ so there is a limit point $(u_*, g_*, p_*) \in H_0^1(\Omega; \mathbb{R}^n) \times Z_{\text{ad}} \times H_0^1(\Omega; \mathbb{R}^n)$ with convergence in appropriate topology (weak convergence of controls)

Proof of asymptotic compatibility (continued)

Step 1: Show that $B_0(u_*, \varphi) \;=\; \langle g_*, \varphi \rangle$ for all $\varphi \in H_0^1(\Omega; \mathbb{R}^n)$
Pick $\varphi \in C_0^\infty(\Omega; \mathbb{R}^n)$, let $w_k := I_k \varphi$ (nodal interpolation), $w_k \to \varphi$ in $W^{1,\infty}(\Omega; \mathbb{R}^n)$, then compute limit as $k \to \infty$ of

$$B_{\delta_k}(\overline{u_k}, w_k) \;=\; \langle A_{\delta_k} \varphi, \overline{u_k} \rangle_{X_0^*, X_0} + \langle A_{\delta_k}(w_k - \varphi), \overline{u_k} \rangle_{X_0^*, X_0} \;=:\; I_k + II_k.$$

Step 2: Show that $B_0(\varphi, p_*) \;=\; \langle u_*, \varphi \rangle$ for all $\varphi \in H_0^1(\Omega; \mathbb{R}^n)$
Strategy is identical to Step 1!

Step 3: Show that $g_*(x) \;=\; \mathbb{P}_{Z_{\text{ad}}} \left(-\frac{1}{\lambda} p_*(x) \right)$
Recall that $\overline{g_k}(x) \;=\; \mathbb{P}_{Z_{\text{ad}}} \left(-\frac{1}{\lambda} \Pi_0 \overline{p_k}(x) \right)$, show $\Pi_0 \overline{p_k} \to p_*$ strongly in $L^2(\Omega; \mathbb{R}^n)$

Proof of asymptotic compatibility (continued)

Step 4: Unraveling

- Steps 1-3 and uniqueness of solutions to optimality system give $\overline{u} = u_*$, $\overline{g} = g_*$, and $\overline{p} = p_*$
- This is the limit point reached for any sub-sequence of original sequence
- Entire sequence of triples $\{(\overline{u_k}, \overline{g_k}, \overline{p_k})\}_{k=1}^{\infty}$ converges to $(\overline{u}, \overline{g}, \overline{p})$

Step 5: Strong convergence of controls
Use the Lipschitz property of the projection and the estimate

$$\|\overline{g_k} - \overline{g}\|_{L^2(\Omega;\mathbb{R}^n)} \lesssim \|\overline{p} - \Pi_0\overline{p}\|_{L^2(\Omega;\mathbb{R}^n)} + \|\Pi_0\overline{p} - \Pi_0\overline{p_k}\|_{L^2(\Omega;\mathbb{R}^n)}$$

Recap and closing

- Showed existence and uniqueness of minimizers
- Considered behavior as $\delta \to 0^+$
- Discretized via FEMs
- Studied simultaneous limit as $\delta, h \to 0^+$

Thank you! Questions?

Auxiliary lemmas

Lemma (Regularity of Control for Fractional-Type Kernels)

Suppose that

$$\frac{c}{|\xi|^{n+2s}} \leq \frac{k_\delta(\xi)}{|\xi|^2} \leq \frac{C}{|\xi|^{n+2s}}$$

holds for all $\xi \in B(0,\delta)$, *for some* $s \neq \frac{1}{2}$. *Then necessarily* $\overline{g_\delta} \in X(\Omega_\delta;\mathbb{R}^n)$.

Recall Fractional Sobolev Space (special case of our kernels):

$$H^s(\Omega_\delta;\mathbb{R}^n) := \left\{ u|_\Omega \in L^2(\Omega;\mathbb{R}^n), \frac{|u(x)-u(y)|}{|x-y|^{\frac{n}{2}+s}} \in L^2(\Omega_\delta \times \Omega_\delta) \right\}$$

Proof strategy:

- Mengesha-Du 2016 says that $H^s(\Omega_\delta;\mathbb{R}^n) = X(\Omega_\delta;\mathbb{R}^n)$ here
- Use projection formula $\overline{g_\delta}(x) = -\frac{1}{\lambda}\mathbb{P}_{Z_{\text{ad}}}(\overline{p_\delta}(x))$
- This pointwise projection is continuous in H^s semi-norm but not X semi-norm!

Auxiliary lemmas (continued)

Lemma (Approximation)

If $w \in L^2(\Omega_\delta; \mathbb{R}^n)$, *then*

$$\|\Pi_0 w - w\|_{L^2(\Omega;\mathbb{R}^n)} \leq \ \omega(h),$$

where $\Pi_0 : L^2(\Omega_\delta; \mathbb{R}^n) \to Z_h$ *denotes the projection of a function onto the piecewise constants with respect to the given mesh. If in fact* $w \in X(\Omega_\delta; \mathbb{R}^n)$ *and* k_δ *satisfies the fractional inequality on* $B(0, \delta)$ *for some* $s \neq \frac{1}{2}$, *then*

$$\|\Pi_0 w - w\|_{L^2(\Omega;\mathbb{R}^n)} \lesssim \ h^s \|w\|_{X(\Omega_\delta;\mathbb{R}^n)}.$$

Back

Control convergence (continued)

Galerkin Approximations:

$q_{\delta,h} \in X_{\delta,h}$ be the Galerkin approximation to $\overline{p_\delta}$, i.e., the solution of

$$B_\delta(v_{\delta,h}, q_{\delta,h}) \;=\; \langle \overline{u_\delta}, v_{\delta,h} \rangle \qquad \forall v_{\delta,h} \in X_{\delta,h}.$$

$U_{\delta,h} \in X_{\delta,h}$ for $\overline{u_\delta}$:

$$B_\delta(U_{\delta,h}, v_{\delta,h}) \;=\; \langle \overline{g_\delta}, v_{\delta,h} \rangle \qquad \forall v_{\delta,h} \in X_{\delta,h}.$$

$r_{\delta,h} \in X_{\delta,h}$ solves

$$B_\delta(v_{\delta,h}, r_{\delta,h}) \;=\; \langle U_{\delta,h}, v_{\delta,h} \rangle \qquad \forall v_{\delta,h} \in X_{\delta,h}.$$

Back

Control convergence (continued)

Let $I_1 := \langle \overline{p_\delta} - \overline{p_{\delta,h}}, \overline{g_{\delta,h}} - \overline{g_\delta} \rangle$ and $I_2 := \langle \overline{p_{\delta,h}} + \lambda \overline{g_{\delta,h}}, \Pi_0 \overline{g_\delta} - \overline{g_\delta} \rangle$. Using optimality conditions gives

$$\lambda \| \overline{g_\delta} - \overline{g_{\delta,h}} \|^2_{L^2(\Omega;\mathbb{R}^n)} \leq \; I_1 + I_2.$$

$$I_1 \; = \; \langle \overline{p_\delta} - q_{\delta,h}, \overline{g_{\delta,h}} - \overline{g_\delta} \rangle + \langle q_{\delta,h} - r_{\delta,h}, \overline{g_{\delta,h}} - \overline{g_\delta} \rangle + \langle r_{\delta,h} - \overline{p_{\delta,h}}, \overline{g_{\delta,h}} - \overline{g_\delta} \rangle \; =: \; I_{1,1} + I_{1,2} + I_{1,3}.$$

By use of Galerkin approximations, find that $I_{1,3} \leq 0$ and

$$I_{1,1} \; \lesssim \; \| \overline{g_{\delta,h}} - \overline{g_\delta} \|_{L^2(\Omega;\mathbb{R}^n)} \inf_{v_{\delta,h} \in X_{\delta,h}} [\overline{p_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)};$$

$$I_{1,2} \; \lesssim \; \| \overline{g_{\delta,h}} - \overline{g_\delta} \|_{L^2(\Omega;\mathbb{R}^n)} \inf_{v_{\delta,h} \in X_{\delta,h}} [\overline{u_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)}.$$

Control convergence (continued)

By Young's Inequality,

$$
\begin{aligned}
I_1 \ \le\ & \frac{\lambda}{3}\|\overline{g_{\delta,h}} - \overline{g_\delta}\|^2_{L^2(\Omega;\mathbb{R}^n)} + C\left(\inf_{v_{\delta,h}\in X_{\delta,h}} [\overline{u_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)}\right)^2 + \\
& C\left(\inf_{v_{\delta,h}\in X_{\delta,h}} [\overline{p_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)}\right)^2 .
\end{aligned}
$$

Control convergence (continued)

Now we write I_2 as

$$\langle \overline{p_{\delta,h}} + \lambda \overline{g_{\delta,h}}, \Pi_0 \overline{g_\delta} - \overline{g_\delta} \rangle = \langle \overline{p_\delta} + \lambda \overline{g_\delta}, \Pi_0 \overline{g_\delta} - \overline{g_\delta} \rangle + \lambda \langle \overline{g_{\delta,h}} - \overline{g_\delta}, \Pi_0 \overline{g_\delta} - \overline{g_\delta} \rangle + \langle \overline{p_{\delta,h}} - r_{\delta,h}, \Pi_0 \overline{g_\delta} - \overline{g_\delta} \rangle + \langle r_{\delta,h} - q_{\delta,h}, \Pi_0 \overline{g_\delta} - \overline{g_\delta} \rangle + \langle q_{\delta,h} - \overline{p_\delta}, \Pi_0 \overline{g_\delta} - \overline{g_\delta} \rangle =: I_{2,1} + I_{2,2} + I_{2,3} + I_{2,4} + I_{2,5}.$$

Control convergence (continued)

Use that $\Pi_0(\overline{p_\delta} + \lambda \overline{g_\delta}) = 0$ to estimate $I_{2,1}$ as

$$I_{2,1} \ \leq \ \omega(h).$$

For $I_{2,2}$, use Cauchy and stability:

$$I_{2,2} \ \leq \ \frac{\lambda}{3} \| \overline{g_{\delta,h}} - \overline{g_\delta} \|^2_{L^2(\Omega;\mathbb{R}^n)} + \omega(h),$$

Control convergence (continued)

For $I_{2,3}$, use Galerkin approximations and the stabilty:

$$I_{2,3} \;\leq\; \omega(h)^2 + \left(\inf_{v_{\delta,h} \in X_{\delta,h}} [\overline{u_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)} \right)^2$$

For $I_{2,4}$ and $I_{2,5}$, use Ceá's lemma and Cauchy:

$$I_{2,4} \;\leq\; \omega(h)^2 + \left(\inf_{v_{\delta,h} \in X_{\delta,h}} [\overline{u_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)} \right)^2 ;$$

$$I_{2,5} \;\leq\; \omega(h)^2 + \left(\inf_{v_{\delta,h} \in X_{\delta,h}} [\overline{p_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)} \right)^2 .$$

Control convergence (continued)

Use Young's Inequality and combine all the estimates to get

$$\frac{\lambda}{3}\|\overline{g_\delta} - \overline{g_{\delta,h}}\|^2_{L^2(\Omega;\mathbb{R}^n)} \lesssim \omega(h)^2 + \left(\inf_{v_{\delta,h}\in X_{\delta,h}} [\overline{u_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)}\right)^2 + \left(\inf_{v_{\delta,h}\in X_{\delta,h}} [\overline{p_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)}\right)^2.$$

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