Asymptotic Compatibility of Optimal Control Problems in Peridynamics

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Problem statement

Find $(\overline{u_\delta}, \overline{g_\delta}) \in X_0 \times Z_{\text{ad}}$ such that

$$I(\overline{u_\delta}, \overline{g_\delta}) \; = \; \min_{g_\delta \in L^2, u_\delta \in X_0} \left\{ \int_\Omega F(x, u_\delta(x))dx + \frac{\lambda}{2} \int_\Omega \Gamma(x)|g_\delta(x)|^2 dx \right\},$$

subject to the strongly coupled system of equations

$$\mathcal{L}_\delta u_\delta \; = \; g_\delta$$

- $\delta \geq 0$ is the degree of non-locality (the **horizon parameter** in peridynamics);
- $\overline{g_\delta}$ is a [vector-valued] external force;
- $\overline{u_\delta}$ represents the [vector-valued] displacement.
- $\mathcal{L}_\delta$ is a non-local operator, exact formula to be given.

Goals

- Show existence and uniqueness of minimizers
- Consider behavior as $\delta \to 0^+$
- Discretize via FEMs
- Study simultaneous limit as $\delta, h \to 0^+$ (asymptotic compatibility)

Bilinear forms

Bilinear form induces non-local state equation:

$$B_\delta(u,v) \;:=\; \iint_{\mathcal{D}_\delta} \mathfrak{A}(x,y)k_\delta(x-y)\frac{Du(x,y)}{|x-y|}\frac{Dv(x,y)}{|x-y|}dxdy \;=\; \int_\Omega g(x)\cdot v(x)dx$$

- $\mathfrak{A}$ is positive, bounded, symmetric
- k_δ a kernel with suitable properties
- Du is projected difference, $Du(x,y) := (u(x)-u(y))\cdot\frac{x-y}{|x-y|}$

State equation

Our function space is based on B_δ :

$$X(\Omega_\delta;\mathbb{R}^n) := \{u|_\Omega \in L^2(\Omega;\mathbb{R}^n), B_\delta(u,u) < \infty\}$$

Version with zero non-local boundary data:

$$X_0(\Omega_\delta;\mathbb{R}^n) := \{u \in X(\Omega_\delta;\mathbb{R}^n), u = 0 \text{ in } \Omega_\delta \setminus \Omega\}$$

Cost functional assumptions

$$I(u,g) := \int_\Omega F(x,u(x))dx + \frac{\lambda}{2}\|g\|^2_{L^2(\Omega;\mathbb{R}^n)}$$

Here Z_{ad} is a box constraint, and $F : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is such that:

1. For all $v \in \mathbb{R}^n$ the mapping $x \mapsto F(x,v)$ is measurable;
2. For all $x \in \Omega$ the mapping $v \mapsto F(x,v)$ is continuous and convex;
3. There exist $c_1 > 0$ and $\ell \in L^1(\Omega)$

$$|F(x,v)| \leq c_1|v|^2 + \ell(x)$$

for all $x \in \Omega$, $v \in \mathbb{R}^n$. Moreover, $\lambda \geq 0$

Well-posedness of optimal control problem

Theorem (Well-posedness)

For any $\delta > 0$: there exists $(\overline{u_\delta}, \overline{g_\delta}) \in X_0(\Omega_\delta; \mathbb{R}^n) \times Z_{ad}$ minimizing

$$I(u_\delta, g_\delta) = \int_\Omega F(x, u_\delta(x))dx + \frac{\lambda}{2}\|g_\delta\|^2_{L^2(\Omega;\mathbb{R}^n)}$$

where $\overline{u_\delta} \in X_0$ solves

$$B_\delta(u_\delta, v_\delta) = \int_\Omega g_\delta(x) \cdot v_\delta(x)dx \quad \forall v_\delta \in X_0$$

Furthermore, if F is strictly convex or $\lambda > 0$, then the minimizer is unique.

Convergence of state equation as $\delta \to 0^+$

Local bilinear form (of Navier-Lamé system of linear elasticity)

$$B_0(u,v) \ := \ C(n)\int_\Omega \mathfrak{a}(x)(2\langle \mathrm{Sym}(\triangledown u), \mathrm{Sym}(\triangledown v)\rangle_F + \mathrm{div}(u)\mathrm{div}(v))dx$$

where $C(n) := \frac{1}{(n+2)(n+4)}$.

Theorem

Suppose $\{(\overline{u_\delta}, \overline{g_\delta})\}_{\delta>0}$ is the family of solutions to the non-local problem. Then, there is $(\overline{u}, \overline{g})$ such that $\overline{u_\delta} \to \overline{u}$ in $L^2(\Omega;\mathbb{R}^n)$ and $\overline{g_\delta} \rightharpoonup \overline{g}$ in $L^2(\Omega;\mathbb{R}^n)$. Moreover, $(\overline{u}, \overline{g})$ solves the local optimal control problem.

Local state equation that $(\overline{u}, \overline{g})$ satisfies the constraint:

$$B_0(u,v) \ = \ \int_\Omega g(x)\cdot v(x)dx$$

Discrete Formulation

- Use continuous piecewise linears for states
- Use piecewise constants for controls
- Discrete optimal control problem is well-posed

Non-local problem convergence: summary

Theorem (Convergence)

In the setting of our problems,

$$\|\overline{u_\delta} - \overline{u_{\delta,h}}\|_{X(\Omega_\delta;\mathbb{R}^n)} \lesssim \omega(h) + \inf_{v_{\delta,h}\in X_{\delta,h}} \|\widehat{u_\delta} - v_{\delta,h}\|_{X(\Omega_\delta;\mathbb{R}^n)} + \inf_{v_{\delta,h}\in X_{\delta,h}} [\overline{u_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)} + \inf_{v_{\delta,h}\in X_{\delta,h}} [\overline{p_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)};$$

$$\|\overline{g_\delta} - \overline{g_{\delta,h}}\|^2_{L^2(\Omega;\mathbb{R}^n)} \lesssim \omega(h)^2 + \left(\inf_{v_{\delta,h}\in X_{\delta,h}} [\overline{u_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)}\right)^2 + \left(\inf_{v_{\delta,h}\in X_{\delta,h}} [\overline{p_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)}\right)^2.$$

$$\|\overline{p_\delta} - \overline{p_{\delta,h}}\|_{X(\Omega_\delta;\mathbb{R}^n)} \lesssim \omega(h) + \inf_{v_{\delta,h}\in X_{\delta,h}} \|\widehat{p_h} - v_{\delta,h}\|_{X(\Omega_\delta;\mathbb{R}^n)} + \inf_{v_{\delta,h}\in X_{\delta,h}} \|\widehat{u_\delta} - v_{\delta,h}\|_{X(\Omega_\delta;\mathbb{R}^n)} + \inf_{v_{\delta,h}\in X_{\delta,h}} [\overline{u_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)} + \inf_{v_{\delta,h}\in X_{\delta,h}} [\overline{p_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)}.$$

What is asymptotic compatibility?

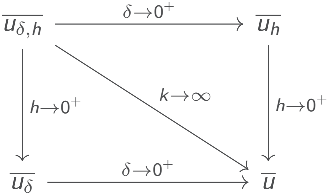
- Introduced by X. Tian and Q. Du (2014)
- Originally developed for the linear, non-local state equation

$$\mathcal{L}_{\delta,h} u_{\delta,h} = f$$

- Unconditional convergence of approximations in both discretization and horizon parameters

Definition (Asymptotic Compatibility)

Given fixed data f in a Hilbert Space, the family of solutions $\{u_{\delta,h}\}_{\delta,h>0}$ is **asymptotically compatible** in $\delta, h > 0$ if for any sequences $\{\delta_k\}_{k=1}^{\infty}, \{h_k\}_{k=1}^{\infty}$ with $\delta_k, h_k \to 0$, we have that $u_{\delta_k,h_k} \to u_0$ strongly in some Hilbert space norm, where u_0 is the solution to a local, continuous problem.

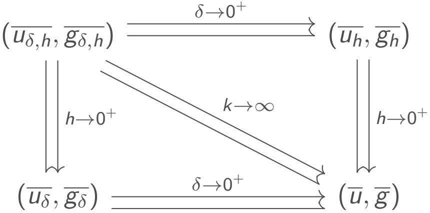


What is asymptotic compatibility? (continued)

Definition (Asymptotic Compatibility for Optimal Control)

We say that the family of solutions $\{(\overline{u_{\delta,h}}, \overline{g_{\delta,h}})\}_{h>0,\delta>0}$ to the nonlocal discrete optimal control problem is **asymptotically compatible** in $\delta, h > 0$ if for any sequences $\{\delta_k\}_{k=1}^{\infty}, \{h_k\}_{k=1}^{\infty}$ with $\delta_k, h_k \to 0$, we have that $\overline{g_{\delta_k,h_k}} \rightharpoonup \overline{g}$ weakly in $L^2(\Omega;\mathbb{R}^n)$, and $\overline{u_{\delta_k,h_k}} \to \overline{u}$ strongly in $L^2(\Omega;\mathbb{R}^n)$. Here $(\overline{u}, \overline{g}) \in H_0^1(\Omega;\mathbb{R}^n) \times Z_{\mathrm{ad}}$ denotes the optimal solution for the local continuous problem.

NOTE: In particular, $H_0^1(\Omega;\mathbb{R}^n)$ is the limiting space of $X_0(\Omega_\delta;\mathbb{R}^n)$ as $\delta \to 0^+$



Asymptotic compatibility: result and notation

Theorem

Assume $\mathfrak{A}$ is Lipschitz. Then, our family of optimal control problems is asymptotically compatible as $\delta, h \to 0^+$.

Define $A_\delta : X_0(\Omega_\delta;\mathbb{R}^n) \to (X_0(\Omega_\delta;\mathbb{R}^n))^*$ as the invertible operator such that

$$\langle A_\delta u, v\rangle_{X_0^*,X_0} \;=\; B_\delta(u,v) \quad \forall u, v \in X_0(\Omega_\delta;\mathbb{R}^n);$$

define $A_0 : H_0^1(\Omega;\mathbb{R}^n) \to H^{-1}(\Omega;\mathbb{R}^n)$ as the invertible operator such that

$$\langle A_0 u, v\rangle_{H^{-1},H_0^1} \;=\; B_0(u,v) \quad \forall u, v \in H_0^1(\Omega;\mathbb{R}^n).$$

Proof of asymptotic compatibility

Step 0: Pick sub-sequence $\{(\overline{u_k}, \overline{g_k}, \overline{p_k})\}_{k=1}^{\infty} \to (u_*, g_*, p_*)$, a triple in $H_0^1(\Omega; \mathbb{R}^n) \times Z_{\text{ad}} \times H_0^1(\Omega; \mathbb{R}^n)$

Step 1: Show that $B_0(u_*, \varphi) \ = \ \langle g_*, \varphi \rangle$ for all $\varphi \in H_0^1(\Omega; \mathbb{R}^n)$
Pick $\varphi \in C_0^\infty(\Omega; \mathbb{R}^n)$, let $w_k := I_k \varphi$ (nodal interpolation), $w_k \to \varphi$ in $W^{1,\infty}(\Omega; \mathbb{R}^n)$, then compute limit as $k \to \infty$ of

$$B_{\delta_k}(\overline{u_k}, w_k) \ = \ \langle A_{\delta_k} \varphi, \overline{u_k} \rangle_{X_0^*, X_0} + \langle A_{\delta_k}(w_k - \varphi), \overline{u_k} \rangle_{X_0^*, X_0} \ =: \ I_k + II_k.$$

Step 2: Show that $B_0(\varphi, p_*) \ = \ \langle u_*, \varphi \rangle$ for all $\varphi \in H_0^1(\Omega; \mathbb{R}^n)$
Strategy is identical to Step 1!

Proof of asymptotic compatibility (continued)

Step 3: Show that $g_*(x) \;=\; \mathbb{P}_{Z_{\text{ad}}}\left(-\frac{1}{\lambda} p_*(x)\right)$
Recall that $\overline{g_k}(x) \;=\; \mathbb{P}_{Z_{\text{ad}}}\left(-\frac{1}{\lambda}\Pi_0\overline{p_k}(x)\right)$, show $\Pi_0\overline{p_k} \to p_*$ strongly in $L^2(\Omega;\mathbb{R}^n)$

Step 4: Unraveling

- Steps 1-3 and uniqueness of solutions to optimality system give $\overline{u} = u_*$, $\overline{g} = g_*$, and $\overline{p} = p_*$
- This is the limit point reached for any sub-sequence of original sequence
- Entire sequence of triples $\{(\overline{u_k}, \overline{g_k}, \overline{p_k})\}_{k=1}^{\infty}$ converges to $(\overline{u}, \overline{g}, \overline{p})$

Recap and closing

- Showed existence and uniqueness of minimizers
- Considered behavior as $\delta \to 0^+$
- Discretized via FEMs
- Studied simultaneous limit as $\delta, h \to 0^+$

Thank you! Questions?