An Optimal Control Problem in Peridynamics: Analysis and Discretization

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Outline

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Problem Statement

Find $(\overline{u_\delta}, \overline{g_\delta}) \in X_0 \times Z_{\text{ad}}$ such that

$$I(\overline{u_\delta}, \overline{g_\delta}) \; = \; \min_{g_\delta \in L^2, u_\delta \in X_0} \left\{ \int_\Omega F(x, u_\delta(x)) dx + \frac{\lambda}{2} \int_\Omega \Gamma(x) |g(x)|^2 dx \right\},$$

over pairs $(u_\delta, g_\delta) \in X_0 \times Z_{\text{ad}}$ that satisfy some state equation (exact form TBD)

$$B_\delta(u_\delta, w_\delta) \; = \; \int_\Omega g_\delta(x) \cdot w_\delta(x), \; \forall w_\delta \in X_0.$$

where $\delta \geq 0$ is the degree of non-locality. Here $\overline{g_\delta}$ is an external force and $\overline{u_\delta}$ represents the displacement

Sample Candidate integrand

Example:

$$F(x, u(x)) = |u(x) - u_{\text{des}}(x)|^2$$

where u_{des} is the optimal shape of the material in space to fit a pre-determined hole as closely as possible

Goals

- Show existence and uniqueness of minimizers
- Consider behavior as $\delta \to 0^+$
- Discretize via FEA
- Study simultaneous limit as $\delta, h \to 0^+$ (asymptotic compatibility)

Outline

Motivation and Origins

For our problem the nonlocal operator is

$$\mathcal{L}_\delta u(x) \;=\; \frac{1}{2}\int_{\Omega_\delta} \mathfrak{A}(x,y)k_\delta(|x-y|)\frac{Du(x,y)}{|x-y|}\frac{y-x}{|x-y|}dy$$

Nonlocal equations [or systems] take the form

$$\begin{cases}\mathcal{L}_\delta u \;=\; g, x \in \Omega \\ u \;=\; 0, x \in \Omega_\delta \setminus \Omega\end{cases}$$

Common in solid state mechanics, including peridynamics

Motivation and Origins (continued)

Definition (PD)

Peridynamics (PD) is a nonlocal model for elasticity of solids that uses integrals over derivatives, attributed to Stewart A. Silling

Features:

- Exchanges derivatives in continuum models for integrals (helps address crack formation)
- Treats particles as having a bond between them (**bond-based model**)
- Range of interaction parameterized by δ, called **horizon**
- Material parameters represented by $\mathfrak{a}(x)$ (e.g., density)
- Operator is elliptic (not parabolic or hyperbolic)

Notation

- Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $\Omega_\delta := \Omega \cup \{x, \text{dist}(x, \partial\Omega) < \delta\}$
- $\Omega_\delta \setminus \Omega$ is non-local boundary
- $\mathcal{D}_\delta := (\Omega \times \Omega_\delta) \cup (\Omega_\delta \times \Omega)$
- Projected difference: $Du(x,y) := \frac{(u(x)-u(y))\cdot(x-y)}{|x-y|}$, nonlocal linearized strain (for vector-valued functions)
- Our material coefficient function is

$$\mathfrak{A}(x,y) \; := \; \frac{\mathfrak{a}(x) + \mathfrak{a}(y)}{2},$$

where there exist $a_{\min}, a_{\max} > 0$ so $a_{\min} \leq \mathfrak{a} \leq a_{\max}$ on Ω_δ.

Properties of Kernels

Kernel sequence $\{k_\delta\}_{\delta>0}$ radial, integrable, non-negative, supported in $B(0,\delta)$, $k_\delta(r)r^{-2}$ is nonincreasing, and

$$\lim_{\delta\to 0^+}\int_{\mathbb{R}^n} k_\delta(\xi)d\xi \;=\; \delta_0$$

Also, for all $\delta > 0$:

$$\int_{\mathbb{R}^n} k_\delta(\xi)d\xi \;=\; 1$$

$$\lim_{t\to\infty}\int_{\mathbb{R}^n\setminus B(0,t)} k_\delta(\xi)d\xi = 0$$

Bi-linear forms

Nonlocal bi-linear form:

$$B_\delta(u,v) \ := \ \frac{1}{2}\iint_{\mathcal{D}_\delta} \mathfrak{A}(x,y)k_\delta(x-y)\frac{Du(x,y)}{|x-y|}\frac{Dv(x,y)}{|x-y|}dxdy$$

Local bi-linear form:

$$B_0(u,v) \ := \ C(n)\int_\Omega \mathfrak{a}(x)(2\langle \mathrm{Sym}(\nabla u), \mathrm{Sym}(\nabla v)\rangle_F + \mathrm{div}(u)\mathrm{div}(v))dx,$$

with $C(n) = \frac{1}{(n+2)(n+4)}$

Inner products denoted $\langle \cdot,\cdot\rangle_Y$; L^2-inner product denoted $\langle \cdot,\cdot\rangle$

Function Spaces

Our function space is based on B_δ :

$$X(\Omega_\delta;\mathbb{R}^n) := \{u|_\Omega \in L^2(\Omega;\mathbb{R}^n), B_\delta(u,u) < \infty\}$$

Version with zero non-local boundary data:

$$X_0(\Omega_\delta;\mathbb{R}^n) := \{u \in X(\Omega_\delta;\mathbb{R}^n), u = 0 \text{ in } \Omega_\delta \setminus \Omega\}$$

Cost Functional Assumptions

$$I(u,g) \ := \ \int_\Omega F(x,u(x))dx + \frac{\lambda}{2}\int_\Omega \Gamma(x)|g(x)|^2 dx.$$

Here Z_{ad} is a nonempty, closed, convex, and bounded subset of $L^2(\Omega;\mathbb{R}^n)$, taking the form

$$Z_{\text{ad}} = \{z \in L^2(\Omega;\mathbb{R}^n), a \preceq z \preceq b\}$$

Here $[a]_i \leq [b]_i$ for all $i \in \{1,2,\ldots,n\}$ with $a = ([a]_1,\ldots,[a]_n)$ and $b = ([b]_1,\ldots,[b]_n)$ being vector fields in $L^2(\Omega;\mathbb{R}^n)$, $\lambda \geq 0$, and $\Gamma \in L^\infty(\Omega)$ is positive.

Cost Functional Assumptions (continued)

The integrand $F : \Omega \times \mathbb{R}^n \to \mathbb{R}$ possesses the following properties:

1. For all $v \in \mathbb{R}^n$ the mapping $x \mapsto F(x, v)$ is measurable;
2. For all $x \in \Omega$ the mapping $v \mapsto F(x, v)$ is continuous and convex;
3. There exist $c_1 > 0$ and $\ell \in L^1(\Omega)$

$$|F(x, v)| \leq \ c_1 |v|^2 + \ell(x)$$

   for all $x \in \Omega$, $v \in \mathbb{R}^n$.

Admissible Classes

$$\mathcal{A}^\delta := \{(v, f) \in X_0(\Omega_\delta; \mathbb{R}^n) \times Z_{\text{ad}},$$
$$v \text{ solves state system with right hand side } f \text{ for any } w \in X_0(\Omega_\delta; \mathbb{R}^n)\}$$

$$\mathcal{A}^{\text{loc}} := \{(v, f) \in H_0^1(\Omega; \mathbb{R}^n) \times Z_{\text{ad}},$$
$$v \text{ solves local BVP for any } w \in H_0^1(\Omega; \mathbb{R}^n)\}$$

Outline

Structural Properties

- $(X(\Omega_\delta; \mathbb{R}^n), \|\cdot\|_X)$ and $(X_0(\Omega_\delta; \mathbb{R}^n), \|\cdot\|_X)$ are Hilbert
- Can extend $u \in X_0(\Omega_\delta; \mathbb{R}^n)$ by zero to any open set $B \supset \Omega_\delta$, including $\mathbb{R}^n$.
- $H^1(\Omega_\delta; \mathbb{R}^n) \hookrightarrow X(\Omega_\delta; \mathbb{R}^n)$
- $X(\Omega_\delta; \mathbb{R}^n) \Subset L^2(\Omega_\delta; \mathbb{R}^n)$

Nonlocal Poincaré Inequality

Proposition (Nonlocal Poincaré-Korn (Mengesha-Du 2014))

There exists a $\delta_0 > 0$ and a constant $C(\delta_0) > 0$ such that for all $\delta \in (0, \delta_0]$ and $u \in X_0(\Omega_\delta; \mathbb{R}^n)$,

$$\|u\|^2_{L^2(\Omega;\mathbb{R}^n)} \leq \; C(\delta_0) \int_{\Omega_\delta} \int_{\Omega_\delta} \frac{k_\delta(x-y)|Du(x,y)|^2}{|x-y|^2} dxdy.$$

Notes:

- The assumption of $k_\delta(r)r^{-2}$ being non-increasing is needed in the proof
- The projected difference vanishes under infinitesimal rigid displacements

State equation is well-posed!

Theorem (Existence and Uniqueness for State Equation)

For any $g_\delta \in L^2$, there exists a unique $u \in X_0$ such that the state system

$$B_\delta(u_\delta, w_\delta) \; = \; \langle g_\delta, w_\delta \rangle$$

is satisfied for all $w_\delta \in X_0$. Furthermore, we have the stability estimate

$$\|u_\delta\|_{X(\Omega_\delta;\mathbb{R}^n)} \lesssim \; \|g_\delta\|_{X(\Omega_\delta;\mathbb{R}^n)^*}$$

for some constant independent of δ.

Minimization Problem

Goal: find $(\overline{u_\delta}, \overline{g_\delta}) \in X_0 \times L^2$ minimizing

$$I(u_\delta, g_\delta) = \int_\Omega F(x, u_\delta(x))dx + \frac{\lambda}{2}\int_\Omega \Gamma(x)|g_\delta(x)|^2 dx$$

subject to: $\lambda \geq 0$, $g_\delta \in Z_{ad} \subset L^2$ and $(u_\delta, g_\delta) \in X_0 \times L^2$ solving

$$B_\delta(u_\delta, v_\delta) = \int_\Omega g_\delta(x) \cdot v_\delta(x)dx \quad \forall v_\delta \in X_0$$

Well-posedness of optimal control problem

Theorem (Well-posedness)

There exists $(\overline{u_\delta}, \overline{g_\delta}) \in X_0(\Omega_\delta; \mathbb{R}^n) \times Z_{ad}$ *minimizing*

$$I(u_\delta, g_\delta) = \int_\Omega F(x, u_\delta(x))dx + \frac{\lambda}{2}\int_\Omega \Gamma(x)|g_\delta(x)|^2 dx,$$

where $\overline{u_\delta} \in X_0$ *solves*

$$B_\delta(u_\delta, v_\delta) = \int_\Omega g_\delta(x) \cdot v_\delta(x) dx \quad \forall v_\delta \in X_0$$

Furthermore, if F *is strictly convex or* $\lambda > 0$*, then the minimizer is unique.*

Use compactness to apply direct method

Outline

Local and Nonlocal Energies

Define for $u \in L^2(\Omega; \mathbb{R}^n)$:

$$\widetilde{E_\delta}(u) := \iint_{\mathcal{D}_\delta} \mathfrak{A}(x,y) k_\delta(x-y) \frac{|Du(x,y)|^2}{|x-y|^2} dxdy - \int_\Omega \overline{g_\delta}(x) \cdot u(x) dx;$$
$$\widetilde{E_0}(u) := C(n) \int_\Omega \mathfrak{a}(x)(2\|\mathrm{Sym}(\bigtriangledown u(x))\|_F^2 + \mathrm{div}(u(x))^2) dx - \int_\Omega \overline{g}(x) \cdot u(x) dx,$$

where $C(n) := \frac{1}{(n+2)(n+4)}$, and $\|\cdot\|_F$ is the Fröbenius norm. Take to be $+\infty$ when not well-defined

Minimization of Local Functional

Theorem

Suppose $\{(\overline{u_\delta}, \overline{g_\delta})\}_{\delta>0}$ denotes the sequence of minimizers for the non-local optimal control problem. If $\overline{u_\delta} \to \overline{u}$ strongly in $L^2(\Omega;\mathbb{R}^n)$ and $\overline{g_\delta} \rightharpoonup \overline{g}$ weakly in $L^2(\Omega;\mathbb{R}^n)$, then $(\overline{u}, \overline{g})$ is the minimizer to the local optimal control problem.

Notice $\{\overline{u}_\delta\}_{\delta>0}$ have bounded semi-norm so compactness gives $\overline{u}$
Notice $\{\overline{g}_\delta\}_{\delta>0}$ are bounded in $L^2(\Omega;\mathbb{R}^n)$ so reflexivity gives $\overline{g}$
Need minimizers to be preserved in limit!

Γ-Convergence

Definition

We say that the family $\widetilde{E_\delta} : L^2(\Omega;\mathbb{R}^n) \to \mathbb{R} \cup \{+\infty\}$ **Γ-converges** strongly in $L^2(\Omega;\mathbb{R}^n)$ to $\widetilde{E_0} : L^2(\Omega;\mathbb{R}^n) \to \mathbb{R} \cup \{+\infty\}$ (denoted $\widetilde{E_\delta} \xrightarrow{\Gamma} \widetilde{E_0}$) if:

i) **The liminf inequality:** Assume $u_\delta \to u$ strongly in $L^2(\Omega;\mathbb{R}^n)$. Then

$$\widetilde{E_0}(u) \le \operatorname{liminf}_{\delta\to 0^+} \widetilde{E_\delta}(u_\delta)$$

ii) **Recovery sequence property:** For each $u \in L^2(\Omega;\mathbb{R}^n)$, there exists a sequence $\{u_\delta\}_{\delta>0}$ where $u_\delta \to u$ strongly in $L^2(\Omega;\mathbb{R}^n)$ and

$$\operatorname{limsup}_{\delta\to 0^+} \widetilde{E_\delta}(u_\delta) \le \widetilde{E_0}(u)$$

Highlights: Recovery Sequence Proof

Use Taylor on $\widetilde{E_\delta}(u)$ and symmetry of $\mathfrak{A}$, focus on controlling

$$C \int_{\Omega_\delta} \int_{\Omega_\delta} \mathfrak{a}(y) k_\delta(x-y)|x-y| dxdy;$$

$$\int_{\Omega_\delta} \int_{\Omega_\delta} \mathfrak{a}(y) k_\delta(x-y) \left\langle \mathrm{Sym}(\triangledown u(x)) \frac{x-y}{|x-y|}, \frac{x-y}{|x-y|} \right\rangle^2 dxdy$$

For first integral: decays to 0 due to support of k_δ and boundedness of Ω
Handle second integral with Fubini, change of variables
Recovery sequence is $u_\delta := u$!

Needed Compactness Result

Proposition (Mengesha-Du 2014)

Suppose $\{u_\delta\}_{\delta>0} \subset L^2(\Omega;\mathbb{R}^n)$ is a bounded family of vector fields such that

$$\sup_{\delta>0} \int_{\Omega_\delta} \int_{\Omega_\delta} k_\delta(x-y) \frac{|Du_\delta(x,y)|^2}{|x-y|^2} dxdy \ < \ \infty,$$

where $k_\delta(r) r^{-2}$ is nonincreasing; then the family has compact closure in $L^2(\Omega;\mathbb{R}^n)$, and any limit point u belongs to $H^1(\Omega_\delta;\mathbb{R}^n)$

In fact, we can show $u \in H^1_0(\Omega;\mathbb{R}^n)$!

Highlights: Lim-inf Inequality Proof

Goal:

$$\widetilde{E_0}(u) \leq \liminf_{\delta\to 0^+} \widetilde{E_\delta}(u_\delta)$$

- Assume without loss of generality $\liminf_{\delta\to 0^+} \widetilde{E_\delta}(u_\delta) < \infty$
- Use compactness, conclude limit point $u \in H_0^1(\Omega; \mathbb{R}^n)$
- Prove liminf inequality for the following forms of $\mathfrak{a}$: indicator functions; simple functions; non-negative $L^\infty(\Omega)$ functions

Preservation of Minimizers

Lemma

If $\{v_\delta\}_{\delta>0}$ *is a sequence of minimizers for* $\{\widetilde{E_\delta}\}_{\delta>0}$ *over* $L^2(\Omega;\mathbb{R}^n)$, v *is a limit point of this sequence, and* $\widetilde{E_\delta} \xrightarrow{\Gamma} \widetilde{E_0}$, *then* v *is a minimizer of* $\widetilde{E_0}$ *on* $L^2(\Omega;\mathbb{R}^n)$. *Finally,*

$$\lim_{\delta\to 0^+} \widetilde{E_\delta}(v_\delta) = \widetilde{E_0}(v)$$

Tying Together Optimal Control

Since $(0, \overline{g_\delta}) \in \mathcal{A}^\delta$ for each $\delta > 0$, we have $\widetilde{E_\delta}(\overline{u}_\delta) \leq 0$; rearrange and use nonlocal Poincaré to get

$$[\overline{u}_\delta]_{X(\Omega_\delta;\mathbb{R}^n)} \ \leq \ C\|\overline{g}_\delta\|_{L^2(\Omega;\mathbb{R}^n)} \leq \ M$$

Finally, for any $(v, f) \in \mathcal{A}^{\text{loc}}$,

$$I(\overline{u}, \overline{g}) \ \leq \ \lim_{\delta \to 0^+} I(\overline{u}_\delta, \overline{g}_\delta) \ \leq \ \lim_{\delta \to 0^+} I(f, v_\delta) \ \leq \ I(f, v),$$

where (v_δ, f) are solutions to the non-local state equation.

Outline

Non-local discrete problem statement

Find $(\overline{u_{\delta,h}}, \overline{g_{\delta,h}}) \in X_{\delta,h} \times Z_h$ such that

$$I(\overline{u_{\delta,h}}, \overline{g_{\delta,h}}) = \min_{u_{\delta,h} \in X_{\delta,h},\ g_{\delta,h} \in Z_h} I(u_{\delta,h},\ g_{\delta,h}),$$

over pairs $(u_{\delta,h}, g_{\delta,h}) \in X_{\delta,h} \times Z_h$ that satisfy

$$B_\delta(u_{\delta,h}, v_{\delta,h}) \ = \ \langle g_{\delta,h}, v_{\delta,h} \rangle, \qquad \forall v_{\delta,h} \in X_{\delta,h}.$$

Recap:

$$I(u_{\delta,h}, g_{\delta,h}) \ := \ \int_\Omega F(x, u_{\delta,h}(x))dx + \frac{\lambda}{2} \|g_{\delta,h}\|^2_{L^2(\Omega;\mathbb{R}^n)}$$

Henceforth assume $\Gamma \equiv 1$

Local discrete problem statement

Find $(\overline{u_h}, \overline{g_h}) \in X_h \times Z_h$ such that

$$I(\overline{u_h}, \overline{g_h}) \;=\; \min_{u_h \in X_h,\ g_h \in Z_h} I(u_h, g_h),$$

over pairs $(u_h, g_h) \in X_h \times Z_h$ that satisfy

$$B_0(u_h, v_h) \;=\; \langle g_h, v_h \rangle, \qquad \forall v_h \in Z_h.$$

Notation

- Mesh family: $\{\mathscr{T}_h\}_{h>0}$ (discretizing Ω_δ) shape-regular and quasi-uniform
- Piecewise polynomials of degree m (with respect to our mesh):

$$\mathcal{P}_m(T;\mathbb{R}^n) := \left\{ \sum_{\alpha\in\mathbb{N}_0^n \,:\, \sum_{i=1}^n \alpha_i \leq m} v_\alpha x_1^{\alpha_1}\cdots x_n^{\alpha_n} \;\middle|\; v_\alpha \in \mathbb{R}^n,\ (x_i)_{i=1}^n \in T \right\}$$

- Discretized state space: $X_{\delta,h} := X_h := \{w_h \in C^0(\overline{\Omega_\delta};\mathbb{R}^n) \mid w_h|_T \in \mathcal{P}_1(T;\mathbb{R}^n)\ \forall T \in \mathscr{T}_h, w_h = 0 \text{ on } \Omega_\delta \setminus \Omega\}$
- Discretized control space: $Z_h := \{z_h|_T \in \mathcal{P}_0(T;\mathbb{R}^n)\ \forall T \in \mathscr{T}_h\}$
- $\Pi_0 : Z_{\text{ad}} \to Z_h$ is piecewise constant projection by averages on each triangle

Notation (continued)

Nonlocal discrete control space: $(X_{\delta,h}, \|\cdot\|_X)$
Local discrete control space $(X_h, \|\cdot\|_{H^1})$

$$\mathcal{A}_h^\delta := \{(w_{\delta,h}, f_{\delta,h}) \in X_{\delta,h} \times Z_h, \\ w_{\delta,h} \text{ solves system with right-hand side } f_{\delta,h} \text{ for any } v_{\delta,h} \in X_{\delta,h}\}$$

$$\mathcal{A}_h^{\text{loc}} := \{(w_h, f_h) \in X_h \times Z_h, \\ w_h \text{ solves local system with right-hand side } f_h \text{ for any } v_h \in X_h\}$$

Optimality Conditions Preliminaries

Reduced cost functional:

$$j(g_\delta) \;:=\; \int_\Omega F(x, S_\delta g_\delta(x))dx + \frac{\lambda}{2}\|g_\delta\|^2_{L^2(\Omega;\mathbb{R}^n)}$$

Assume now that

$$F(x, v) := \frac{1}{2}|v|^2.$$

First-order necessary condition:

$$\langle j'(\overline{g_\delta}), \gamma_z - \overline{g_\delta}\rangle \;\geq\; 0 \qquad \forall \gamma_z \in Z_{\text{ad}}$$

Optimality Conditions

Non-local continuous optimality conditions (including adjoint)

$$\begin{aligned} \langle \overline{p_\delta} + \lambda \overline{g_\delta}, \gamma_z - \overline{g_\delta} \rangle &\geq 0, \quad \forall \gamma_z \in Z_{\text{ad}} \\ \overline{p_\delta} = S_\delta^* F_u(\cdot, \overline{u_\delta}) &= S_\delta F_u(\cdot, \overline{u_\delta}) \\ \overline{u_\delta} &= S_\delta \overline{g_\delta}. \end{aligned}$$

Projection formula:

$$\overline{g_\delta}(x) = \mathbb{P}_{Z_{\text{ad}}} \left(-\frac{1}{\lambda} \overline{p_\delta}(x) \right).$$

NOTE: No second-order optimality conditions needed (strict convexity!)

Optimality Conditions (discretized)

Non-local discrete optimality conditions (including adjoint)

$$\begin{aligned}
\langle \overline{p_{\delta,h}} + \lambda \overline{g_{\delta,h}}, \gamma_h - \overline{g_{\delta,h}} \rangle &\geq 0, \quad \forall \gamma_h \in Z_{\text{ad}} \cap Z_h \\
\overline{p_{\delta,h}} = S^*_{\delta,h} F_u(\cdot, \overline{u_{\delta,h}}) &= S_{\delta,h} F_u(\cdot, \overline{u_{\delta,h}}) \\
\overline{u_{\delta,h}} &= S_{\delta,h} \overline{g_{\delta,h}}.
\end{aligned}$$

Projection formula:

$$\overline{g_{\delta,h}}(x) = \mathbb{P}_{Z_{\text{ad}}} \left(-\frac{1}{\lambda} \Pi_0 \overline{p_{\delta,h}}(x) \right)$$

Intermediary Functions

By Lax-Milgram, we may define $\widehat{u_\delta}, \widehat{p_\delta} \in X_0(\Omega_\delta; \mathbb{R}^n)$ such that

$$B_\delta(\widehat{u_\delta}, v_\delta) \;=\; \langle \overline{g_{\delta,h}}, v_\delta \rangle \quad \forall v_\delta \in X_0(\Omega_\delta; \mathbb{R}^n);$$

$$B_\delta(v_\delta, \widehat{p_\delta}) \;=\; \langle v_\delta, \overline{u_{\delta,h}} \rangle \quad \forall v_\delta \in X_0(\Omega_\delta; \mathbb{R}^n);$$

also define $\widehat{u_h}, \widehat{p_h} \in H_0^1(\Omega; \mathbb{R}^n)$ such that

$$B_0(\widehat{u_h}, v) \;=\; \langle \overline{g_h}, v \rangle \quad \forall v \in H_0^1(\Omega; \mathbb{R}^n);$$

$$B_0(v, \widehat{p_h}) \;=\; \langle v, \widehat{u_h} \rangle \quad \forall v \in H_0^1(\Omega; \mathbb{R}^n).$$

Control and State Error Estimate

Theorem (State and Adjoint Error Estimates)

Suppose that $(\overline{u_{\delta,h}}, \overline{g_{\delta,h}})$ *is the solution to the non-local discrete problem,* $\overline{p_{\delta,h}}$ *solves the discrete adjoint equation given* $\overline{u_{\delta,h}}$*;* $(\overline{u_\delta}, \overline{g_\delta})$ *is the solution to the nonlocal continuous problem; and* $\overline{p_\delta}$ *solves the continuous adjoint equation given* $\overline{u_\delta}$*. Then*

$$\|\overline{u_\delta} - \overline{u_{\delta,h}}\|_{X(\Omega_\delta;\mathbb{R}^n)} \lesssim \inf_{v_{\delta,h} \in X_{\delta,h}} \|\widehat{u_\delta} - v_{\delta,h}\|_{X(\Omega_\delta;\mathbb{R}^n)} + \|\overline{g_\delta} - \overline{g_{\delta,h}}\|_{L^2(\Omega;\mathbb{R}^n)};$$

$$\|\overline{p_\delta} - \overline{p_{\delta,h}}\|_{X(\Omega_\delta;\mathbb{R}^n)} \lesssim \inf_{v_{\delta,h} \in X_{\delta,h}} \|\widehat{p_\delta} - v_{\delta,h}\|_{X(\Omega_\delta;\mathbb{R}^n)} + \inf_{v_{\delta,h} \in X_{\delta,h}} \|\widehat{u_\delta} - v_{\delta,h}\|_{X(\Omega_\delta;\mathbb{R}^n)} + \|\overline{g_\delta} - \overline{g_{\delta,h}}\|_{L^2(\Omega;\mathbb{R}^n)}$$

State and Adjoint Error Estimates (continued)

Proof strategy for state error:

- Test with $v_\delta := \overline{u_\delta} - \widehat{u_\delta}$ in state system and intermediary equation, subtract
- Use Hölder Inequality to get $\|\overline{u_\delta} - \widehat{u_\delta}\|_{X(\Omega_\delta;\mathbb{R}^n)} \lesssim \|\overline{g_\delta} - \overline{g_{\delta,h}}\|_{L^2(\Omega;\mathbb{R}^n)}$
- Use Ceá's Lemma on space $X_{\delta,h}$

Note: Adjoint error estimate proven by same strategy

State and Adjoint Error Estimates (local)

Theorem

Let $(\overline{u}, \overline{g})$ denote the solution to the local continuous problem, while $(\overline{u_h}, \overline{g_h})$ is the solution to the local discrete problem. Assume also that $\overline{p}$ denotes the solution to the continuous adjoint problem, while $\overline{p_h}$ solves the discrete adjoint problem. Then,

$$\|\overline{u} - \overline{u_h}\|_{H^1(\Omega;\mathbb{R}^n)} \lesssim \inf_{v_h \in X_h} [\widehat{u_h} - v_h]_{H^1(\Omega;\mathbb{R}^n)} + \|\overline{g} - \overline{g_h}\|_{L^2(\Omega;\mathbb{R}^n)};$$

$$\|\overline{p} - \overline{p_h}\|_{H^1(\Omega;\mathbb{R}^n)} \lesssim \inf_{v_{\delta,h} \in X_h} [\widehat{p_h} - v_h]_{H^1(\Omega;\mathbb{R}^n)} + \inf_{v_h \in X_h} [\widehat{u_h} - v_h]_{H^1(\Omega;\mathbb{R}^n)} + \|\overline{g} - \overline{g_h}\|_{L^2(\Omega;\mathbb{R}^n)}.$$

Control Convergence

Theorem (Convergence of Controls)

Assume that $\overline{g_\delta}$ is the optimal control associated with the nonlocal continuous problem, and $\overline{g_{\delta,h}}$ be the discrete optimal control. Then we have the convergence

$$\|\overline{g_\delta} - \overline{g_{\delta,h}}\|^2_{L^2(\Omega;\mathbb{R}^n)} \lesssim \omega(h) + \left(\inf_{v_{\delta,h} \in X_{\delta,h}} [\overline{u_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)} \right)^2 + \left(\inf_{v_{\delta,h} \in X_{\delta,h}} [\overline{p_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)} \right)^2 .$$

Proof

Non-local Problem Convergence: Summary

Corollary (Full Norm Solution Convergence)

In the setting of our problem formulation,

$$\|\overline{u_\delta} - \overline{u_{\delta,h}}\|_{X(\Omega_\delta;\mathbb{R}^n)} \lesssim \omega(h) + \inf_{v_{\delta,h}\in X_{\delta,h}} \|\widehat{u_\delta} - v_{\delta,h}\|_{X(\Omega_\delta;\mathbb{R}^n)} + \inf_{v_{\delta,h}\in X_{\delta,h}} [\overline{u_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)} + \inf_{v_{\delta,h}\in X_{\delta,h}} [\overline{p_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)};$$

$$\|\overline{p_\delta} - \overline{p_{\delta,h}}\|_{X(\Omega_\delta;\mathbb{R}^n)} \lesssim \omega(h) + \inf_{v_{\delta,h}\in X_{\delta,h}} \|\widehat{p_h} - v_{\delta,h}\|_{X(\Omega_\delta;\mathbb{R}^n)} + \inf_{v_{\delta,h}\in X_{\delta,h}} \|\widehat{u_\delta} - v_{\delta,h}\|_{X(\Omega_\delta;\mathbb{R}^n)} + \inf_{v_{\delta,h}\in X_{\delta,h}} [\overline{u_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)} + \inf_{v_{\delta,h}\in X_{\delta,h}} [\overline{p_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)}.$$

Local Convergence Results

Theorem

Suppose $(\overline{u}, \overline{g})$ *denotes the solution to the local continuous problem, while* $(\overline{u_h}, \overline{g_h})$ *is the solution to the local discrete problem, and* $\overline{p}$ *and* $\overline{p_h}$ *solve the respective adjoint problems. We have the estimates*

$$\|\overline{u} - \overline{u_h}\|_{H^1(\Omega;\mathbb{R}^n)} \lesssim \inf_{v_h \in X_h} [\widehat{u} - v_h]_{H^1(\Omega;\mathbb{R}^n)} + \|\overline{g} - \overline{g_h}\|_{L^2(\Omega;\mathbb{R}^n)};$$

$$\|\overline{p} - \overline{p}\|_{H^1(\Omega;\mathbb{R}^n)} \lesssim \inf_{v_h \in X_h} [\widehat{p} - v_h]_{H^1(\Omega;\mathbb{R}^n)} + \inf_{v_h \in X_h} [\widehat{u} - v_h]_{H^1(\Omega;\mathbb{R}^n)} + \|\overline{g} - \overline{g_h}\|_{L^2(\Omega;\mathbb{R}^n)}.$$

$$\|\overline{g} - \overline{g_h}\|_{L^2(\Omega;\mathbb{R}^n)} \lesssim h + \inf_{v_h \in X_h} [\overline{p} - v_h]_{H^1(\Omega;\mathbb{R}^n)} + \inf_{v_h \in X_h} [\overline{u} - v_h]_{H^1(\Omega;\mathbb{R}^n)}.$$

Discrete Analogues of Convergence

Proposition (Γ-convergence of discrete problems)

We have that $\widetilde{E_\delta} \xrightarrow{\Gamma} \widetilde{E_0}$ in the family of spaces $\{X_{\delta,h}\}_{\delta>0}$ in the strong $L^2(\Omega;\mathbb{R}^n)$ topology.

Theorem (Discrete Convergence)

Suppose $\{(\overline{u_{\delta,h}}, \overline{g_{\delta,h}})\}_{\delta>0} \in \mathcal{A}_h^\delta$ is the family of solutions to the non-local discrete problem. Then, there is $(\overline{u_h}, \overline{g_h}) \in \mathcal{A}_h^{loc}$ such that $\overline{u_{\delta,h}} \to \overline{u_h}$ in $L^2(\Omega;\mathbb{R}^n)$ and $\overline{g_{\delta,h}} \rightharpoonup \overline{g_\delta}$ in $L^2(\Omega;\mathbb{R}^n)$. Moreover, $(\overline{u_h}, \overline{g_h})$ solves the local discrete optimal control problem.

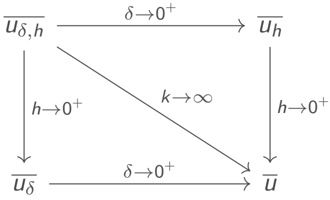
Outline

What is asymptotic compatibility?

- Introduced by X. Tian and Q. Du (2014)
- Originally developed for linear, non-local state equations
- Unconditional convergence of approximations in both discretization and horizon parameters

Definition (Asymptotic Compatibility)

Given fixed data f in a Hilbert Space, the family of solutions $\{u_{\delta,h}\}_{\delta,h>0}$ is **asymptotically compatible** in $\delta, h > 0$ if for any sequences $\{\delta_k\}_{k=1}^{\infty}, \{h_k\}_{k=1}^{\infty}$ with $\delta_k, h_k \to 0$, we have that $u_{\delta_k,h_k} \to u_0$ strongly in some Hilbert space norm, where u_0 is the solution to a local, continuous problem.

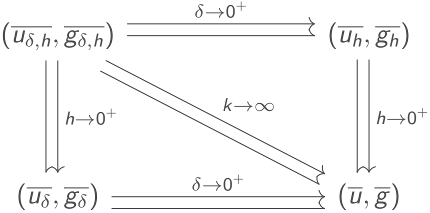


What is asymptotic compatibility? (continued)

Definition (Asymptotic Compatibility for Optimal Control)

We say that the family of solutions $\{(\overline{u_{\delta,h}}, \overline{g_{\delta,h}})\}_{h>0,\delta>0}$ to the nonlocal discrete optimal control problem is **asymptotically compatible** in $\delta, h > 0$ if for any sequences $\{\delta_k\}_{k=1}^{\infty}, \{h_k\}_{k=1}^{\infty}$ with $\delta_k, h_k \to 0$, we have that $\overline{g_{\delta_k,h_k}} \rightharpoonup \overline{g}$ weakly in $L^2(\Omega;\mathbb{R}^n)$, and $\overline{u_{\delta_k,h_k}} \to \overline{u}$ strongly in $L^2(\Omega;\mathbb{R}^n)$. Here $(\overline{u},\overline{g}) \in H_0^1(\Omega;\mathbb{R}^n) \times Z_{\text{ad}}$ denotes the optimal solution for the local continuous problem.

NOTE: In particular, $H_0^1(\Omega;\mathbb{R}^n)$ is the limiting space of $X_0(\Omega_\delta;\mathbb{R}^n)$ as $\delta \to 0^+$



Asymptotic compatibility: result and notation

Theorem

Assume $\mathfrak{A}$ is Lipschitz. Then, our family of optimal control problems is asymptotically compatible as $\delta, h \to 0^+$.

Define $A_\delta : X_0(\Omega_\delta; \mathbb{R}^n) \to (X_0(\Omega_\delta; \mathbb{R}^n))^*$ as the invertible operator such that

$$\langle A_\delta u, v \rangle_{X_0^*, X_0} = B_\delta(u, v) \quad \forall u, v \in X_0(\Omega_\delta; \mathbb{R}^n);$$

define $A_0 : H_0^1(\Omega; \mathbb{R}^n) \to H^{-1}(\Omega; \mathbb{R}^n)$ as the invertible operator such that

$$\langle A_0 u, v \rangle_{H^{-1}, H_0^1} = B_0(u, v) \quad \forall u, v \in H_0^1(\Omega; \mathbb{R}^n).$$

Proof of asymptotic compatibility

Needed preliminaries:

1. Given a $v \in H_0^1(\Omega; \mathbb{R}^n)$, and $h_k, \delta_k \to 0$, we can find a sequence $v_k \in X_{\delta_k, h_k}$ such that $v_k \to v$ strongly in $H^1(\Omega; \mathbb{R}^n)$ as $k \to \infty$.
2. For any sequences $\{\delta_k\}_{k=1}^\infty, \{h_k\}_{k=1}^\infty$ with $\delta_k, h_k \to 0$, there exists a $C > 0$ so that $\|\overline{u_{\delta_k, h_k}}\|_{X(\Omega_{\delta_k}; \mathbb{R}^n)} \leq C$ uniformly in $k \in \mathbb{N}^+$.
3. We have that $A_\delta u \in L^2(\Omega; \mathbb{R}^n)$ and $\lim_{\delta \to 0^+} \|A_\delta u - A_0 u\|_{L^2(\Omega; \mathbb{R}^n)} = 0$ for each $u \in C_0^\infty(\Omega; \mathbb{R}^n)$.

Pick sub-sequence of $\{(\overline{u_k}, \overline{g_k}, \overline{p_k})\}_{k=1}^\infty$ so there is a limit point $(u_*, g_*, p_*) \in H_0^1(\Omega; \mathbb{R}^n) \times Z_{\text{ad}} \times H_0^1(\Omega; \mathbb{R}^n)$ with convergence in appropriate topology

Proof of asymptotic compatibility (continued)

Step 1: Show that $B_0(u_*, \varphi) = \langle g_*, \varphi \rangle$ for all $\varphi \in H_0^1(\Omega; \mathbb{R}^n)$
Pick $\varphi \in C_0^\infty(\Omega; \mathbb{R}^n)$, let $w_k := I_k \varphi$ (nodal interpolation), $w_k \to \varphi$ in $W^{1,\infty}(\Omega; \mathbb{R}^n)$, then compute limit as $k \to \infty$ of

$$B_{\delta_k}(\overline{u_k}, w_k) = \langle A_{\delta_k} \varphi, \overline{u_k} \rangle_{X_0^*, X_0} + \langle A_{\delta_k}(w_k - \varphi), \overline{u_k} \rangle_{X_0^*, X_0} =: I_k + II_k.$$

Step 2: Show that $B_0(\varphi, p_*) = \langle u_*, \varphi \rangle$ for all $\varphi \in H_0^1(\Omega; \mathbb{R}^n)$
Strategy is identical to Step 1!

Step 3: Show that $g_*(x) = -\frac{1}{\lambda} \mathbb{P}_{Z_{\text{ad}}}(p_*(x))$
Recall that $\overline{g_k}(x) = -\frac{1}{\lambda} \mathbb{P}_{Z_{\text{ad}}}(\Pi_0 \overline{p_k}(x))$, show $\Pi_0 \overline{p_k} \to p_*$ strongly in $L^2(\Omega; \mathbb{R}^n)$

Proof of asymptotic compatibility (continued)

Step 4: Unraveling

- Steps 1-3 and uniqueness of solutions to optimality system give $\overline{u} = u_*$, $\overline{g} = g_*$, and $\overline{p} = p_*$
- This is the limit point reached for any sub-sequence of original sequence
- Entire sequence of triples $\{(\overline{u_k}, \overline{g_k}, \overline{p_k})\}_{k=1}^{\infty}$ converges to $(\overline{u}, \overline{g}, \overline{p})$

Recap and closing

- Showed existence and uniqueness of minimizers
- Considered behavior as $\delta \to 0^+$
- Discretized via FEAs
- Studied simultaneous limit as $\delta, h \to 0^+$ (asymptotic compatibility)

Thank you! Questions?

ArXiV preprint: `https://arxiv.org/pdf/2304.09328.pdf`

Auxiliary Lemmas

Lemma (Regularity of Control for Fractional-Type Kernels)

Suppose that

$$\frac{c}{|\xi|^{n+2s}} \leq \frac{k_\delta(\xi)}{|\xi|^2} \leq \frac{C}{|\xi|^{n+2s}}$$

holds for all $\xi \in B(0,\delta)$, for some $s \neq \frac{1}{2}$. Then necessarily $\overline{g_\delta} \in X(\Omega_\delta; \mathbb{R}^n)$.

Proof strategy:

- Mengesha-Du 2016 says that $H^s(\Omega_\delta; \mathbb{R}^n) = X(\Omega_\delta; \mathbb{R}^n)$ here
- Use projection formula $\overline{g_\delta}(x) = -\frac{1}{\lambda}\mathbb{P}_{Z_{\text{ad}}}(\overline{p_\delta}(x))$
- This pointwise projection is continuous in H^s semi-norm but not X semi-norm!

Auxiliary Lemmas (continued)

Lemma (Approximation)

If $w \in L^2(\Omega_\delta; \mathbb{R}^n)$, *then*

$$\|\Pi_0 w - w\|_{L^2(\Omega;\mathbb{R}^n)} \leq \ \omega(h),$$

where $\Pi_0 : L^2(\Omega_\delta; \mathbb{R}^n) \to Z_h$ *denotes the projection of a function onto the piecewise constants with respect to the given mesh. If in fact* $w \in X(\Omega_\delta; \mathbb{R}^n)$ *and* k_δ *satisfies the fractional inequality on* $B(0, \delta)$ *for some* $s \neq \frac{1}{2}$, *then*

$$\|\Pi_0 w - w\|_{L^2(\Omega;\mathbb{R}^n)} \lesssim \ h^s \|w\|_{X(\Omega_\delta;\mathbb{R}^n)}.$$

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Control Convergence (continued)

Galerkin Approximations:

$q_{\delta,h} \in X_{\delta,h}$ be the Galerkin approximation to $\overline{p_\delta}$, i.e., the solution of

$$B_\delta(v_{\delta,h}, q_{\delta,h}) \;=\; \langle \overline{u_\delta}, v_{\delta,h} \rangle \qquad \forall v_{\delta,h} \in X_{\delta,h}.$$

$U_{\delta,h} \in X_{\delta,h}$ for $\overline{u_\delta}$:

$$B_\delta(U_{\delta,h}, v_{\delta,h}) \;=\; \langle \overline{g_\delta}, v_{\delta,h} \rangle \qquad \forall v_{\delta,h} \in X_{\delta,h}.$$

$r_{\delta,h} \in X_{\delta,h}$ solves

$$B_\delta(v_{\delta,h}, r_{\delta,h}) \;=\; \langle U_{\delta,h}, v_{\delta,h} \rangle \qquad \forall v_{\delta,h} \in X_{\delta,h}.$$

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Control Convergence (continued)

Let $I_1 := \langle \overline{p_\delta} - \overline{p_{\delta,h}}, \overline{g_{\delta,h}} - \overline{g_\delta} \rangle$ and $I_2 := \langle \overline{p_{\delta,h}} + \lambda \overline{g_{\delta,h}}, \Pi_0 \overline{g_\delta} - \overline{g_\delta} \rangle$. Using optimality conditions gives

$$\lambda \| \overline{g_\delta} - \overline{g_{\delta,h}} \|^2_{L^2(\Omega;\mathbb{R}^n)} \leq \; I_1 + I_2.$$

$$I_1 \; = \; \langle \overline{p_\delta} - q_{\delta,h}, \overline{g_{\delta,h}} - \overline{g_\delta} \rangle + \langle q_{\delta,h} - r_{\delta,h}, \overline{g_{\delta,h}} - \overline{g_\delta} \rangle + \langle r_{\delta,h} - \overline{p_{\delta,h}}, \overline{g_{\delta,h}} - \overline{g_\delta} \rangle \; =: \; I_{1,1} + I_{1,2} + I_{1,3}.$$

By use of Galerkin approximations, find that $I_{1,3} \leq 0$ and

$$I_{1,1} \; \lesssim \; \| \overline{g_{\delta,h}} - \overline{g_\delta} \|_{L^2(\Omega;\mathbb{R}^n)} \inf_{v_{\delta,h} \in X_{\delta,h}} [\overline{p_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)};$$

$$I_{1,2} \; \lesssim \; \| \overline{g_{\delta,h}} - \overline{g_\delta} \|_{L^2(\Omega;\mathbb{R}^n)} \inf_{v_{\delta,h} \in X_{\delta,h}} [\overline{u_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)}.$$

Control Convergence (continued)

By Young's Inequality,

$$I_1 \le \frac{\lambda}{3}\|\overline{g_{\delta,h}} - \overline{g_\delta}\|^2_{L^2(\Omega;\mathbb{R}^n)} + C\left(\inf_{v_{\delta,h}\in X_{\delta,h}}[\overline{u_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)}\right)^2 + C\left(\inf_{v_{\delta,h}\in X_{\delta,h}}[\overline{p_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)}\right)^2.$$

Control Convergence (continued)

Now we write I_2 as

$$\langle \overline{p_{\delta,h}} + \lambda \overline{g_{\delta,h}}, \Pi_0 \overline{g_\delta} - \overline{g_\delta} \rangle \;=\; \langle \overline{p_\delta} + \lambda \overline{g_\delta}, \Pi_0 \overline{g_\delta} - \overline{g_\delta} \rangle + \lambda \langle \overline{g_{\delta,h}} - \overline{g_\delta}, \Pi_0 \overline{g_\delta} - \overline{g_\delta} \rangle + \langle \overline{p_{\delta,h}} - r_{\delta,h}, \Pi_0 \overline{g_\delta} - \overline{g_\delta} \rangle + \langle r_{\delta,h} - q_{\delta,h}, \Pi_0 \overline{g_\delta} - \overline{g_\delta} \rangle + \langle q_{\delta,h} - \overline{p_\delta}, \Pi_0 \overline{g_\delta} - \overline{g_\delta} \rangle \;=: I_{2,1} + I_{2,2} + I_{2,3} + I_{2,4} + I_{2,5}.$$

Control Convergence (continued)

Use that $\Pi_0(\overline{p_\delta} + \lambda \overline{g_\delta}) = 0$ to estimate $I_{2,1}$ as

$$I_{2,1} \;\leq\; \omega(h).$$

For $I_{2,2}$, use Cauchy and stability:

$$I_{2,2} \;\leq\; \frac{\lambda}{3} \|\overline{g_{\delta,h}} - \overline{g_\delta}\|^2_{L^2(\Omega;\mathbb{R}^n)} + \omega(h),$$

Control Convergence (continued)

For $I_{2,3}$, use Galerkin approximations and the stabilty:

$$I_{2,3} \;\leq\; \omega(h)^2 + \left(\inf_{v_{\delta,h} \in X_{\delta,h}} [\overline{u_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)} \right)^2$$

For $I_{2,4}$ and $I_{2,5}$, use Ceá's lemma and Cauchy:

$$I_{2,4} \;\leq\; \omega(h)^2 + \left(\inf_{v_{\delta,h} \in X_{\delta,h}} [\overline{u_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)} \right)^2 ;$$

$$I_{2,5} \;\leq\; \omega(h)^2 + \left(\inf_{v_{\delta,h} \in X_{\delta,h}} [\overline{p_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)} \right)^2 .$$

Control Convergence (continued)

Use Young's Inequality and combine all the estimates to get

$$\frac{\lambda}{3}\|\overline{g_\delta} - \overline{g_{\delta,h}}\|^2_{L^2(\Omega;\mathbb{R}^n)} \lesssim \omega(h)^2 + \left(\inf_{v_{\delta,h}\in X_{\delta,h}} [\overline{u_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)}\right)^2 + \left(\inf_{v_{\delta,h}\in X_{\delta,h}} [\overline{p_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)}\right)^2.$$

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