How to walk to infinity on primes and square-free walks

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What is a prime walk to infinity?

Question

Can we append digits to a number forever while staying prime?

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1. Do we append digits to the left or right?
2. How many do we append at a time?
3. Do we have to start with a one-digit number?

A Simple Case

Let $\mathscr{P} \subset \mathbb{N}^+$ denote the set of all primes.

Proposition

*If we append arbitrarily many digits at a time to the **left**, then we can walk to infinity.*

Proof details:

- Let $p \in \mathscr{P} \setminus \{2, 5\}$. Pick m large enough so $10^m > p$

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- Let $p \in \mathscr{P} \setminus \{2, 5\}$. Pick m large enough so $10^m > p$
- There are infinitely many primes congruent to p modulo 10^m
- Can pick any sequence of them, append to the left of p

Why these sequences?

Definition (Square-Free Number)

A *square-free number* is an integer that is not divisible by any perfect square other than 1.

- The asymptotic density of the primes less than or equal to x is $\frac{1}{\log(x)} \to 0$ as $x \to \infty$
- However, the square-free numbers have asymptotic density $\frac{6}{\pi^2}$
- So, we are comparing a sequence with zero density to a sequence of positive density! One expects walks on square-free numbers to be "easier"

Computational Challenges

- Tree search (seeing all different possible digits to append)

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Computational Challenges

- Tree search (seeing all different possible digits to append)
- Factorization of large numbers (to check being prime or square-free)
- Primes have an erratic structure

So, we focus on *stochastic* models of walks to infinity and consider sequences with more structure

Game Plan

- Consider random appendages of digits in prime and square free walks
- Consider the walks in different bases
- Will focus on studying the distributions of the stochastic walks, especially expected values
- **Important assumption:** The sequence of numbers appended in a walk are considered independently of each other

Our First Algorithm:

Consider a prime walk in base b. If we are at the stage of having k digits, the probability a random digit is a legal appendage (to the right) is $\frac{1}{k\log(b)}$, so the probability that at least one of the b digits can legally be added is

$$1-\left(1-\frac{1}{k\log b}\right)^{b}.$$

Algorithm (Blind Unlimited Prime Walk in Base b)

Choose one of the possible digits uniformly at random and check if the obtained number is prime; if it is not, stop and record the length; otherwise, continue the process.

Expected Length

Proposition (Blind Unlimited Prime Walk Lengths)

The theoretical expected length of a walk with a starting point at most s digits in base b is

$$E[Y_{s,b}] := \frac{1}{\frac{b^s}{s \log b}} \left(\sum_{r=1}^{s} \frac{(b-1)b^{r-1}}{r \log b} \left(\sum_{n=0}^{\infty} \prod_{k=r}^{n-1} \left(1 - \left(1 - \frac{1}{k \log b} \right)^{b} \right) \right) \right)$$

Analysis of formula:

- The r denotes the number of digits in starting point
- The $\frac{(b-1)b^{r-1}}{r \log b}$ denotes approximate number of primes in base b with r digits
- Divide by $\frac{b^s}{s \log(b)}$, approximate number of primes in base b with at most s digits

Expected Prime Walk Length Data

Expected length of blind unlimited prime walks

Maximum number of starting digits	1	2	3	4	5	6	7
Base 2	5.20	9.90	11.62	11.45	10.40	9.08	7.79
Base 3	5.05	7.75	7.60	6.53	5.40	4.49	3.80
Base 4	4.87	6.55	5.86	4.79	3.92	3.29	2.85
Base 5	4.71	5.79	4.92	3.96	3.25	2.78	2.45
Base 6	4.57	5.27	4.34	3.48	2.89	2.49	2.22
Base 7	4.46	4.89	3.95	3.17	2.65	2.31	2.08
Base 8	4.37	4.59	3.67	2.95	2.49	2.19	1.98
Base 9	4.29	4.36	3.45	2.79	2.37	2.09	1.91
Base 10	4.22	4.17	3.28	2.66	2.28	2.20	1.85

Negative Prime Walks Result

Theorem

It is impossible to walk to infinity on primes in base 2 by appending no more than 2 digits at a time to the right.

Key step: Continuously appending 1 to a prime in base 2 creates a generalized Cunningham chain, which will have consecutive composite numbers. A **Cunningham Chain** is defined via the formula $e_i := 2^i p + 2^i - 1$.

Theorem

It is impossible to walk to infinity on primes in bases $b \in \{3, 4, 5, 6\}$ by appending just 1 digit at a time to the right.

Key step: Use Fermat's Little Theorem repeatedly on sequences of the form $p_i = b^{i-1}p + b^{i-1} - 1$ for some prime p

Why Square-Free Numbers Give Hope

Definition (Square-Free Number)

A *square-free number* is an integer that is not divisible by any perfect square other than 1.

Can easily append digits to 2, pick smallest possible to get square-free at each step:

$$\{2, 21, 210, 2101, 21010, 210101, 2101010, 21010101, 210101010, 2101010101, 21010101010, 210101010101, 2101010101010, 21010101010102, 210101010101021, 2101010101010210, 21010101010102101, \ldots\}.$$

- If $Q(x) := |\{k \in \mathbb{N}^+, k \leq x, k \text{ is square-free}\}|$ then it is known that $Q(x) \sim \frac{6x}{\pi^2}$
- When constructing square-free walks, we say each constructed number is square-free with probability $p := \frac{6}{\pi^2}$

Blind Unlimited Square-Free Walks

Algorithm (Blind Unlimited Square-Free Walk)

Choose one digit uniformly at random from the set $\{0, 1, \ldots, 9\}$ and append it: if the obtained number is not square-free, stop and record the length; otherwise, continue the process.

Experimental expected lengths of the square-free walks in base 10

Start has x digits	0	1	2	3	4	5	6
Blind unlimited square-free walk	1.68	2.79	2.76	2.72	2.71	2.71	2.71

Frequency of Digits

Comparing frequencies of digits of blind unlimited square-free walks in base 10

Number of digits of starting point	1	2	3	4	5	6
Add digit 0	10.1%	7.4%	7.6%	7.5%	7.5%	7.5%
Add digit 1	14.0%	13.6%	13.2%	13.4%	13.4%	13.4%
Add digit 2	8.4%	5.5%	5.3%	5.3%	5.3%	5.3%
Add digit 3	13.5%	13.5%	13.4%	13.4%	13.4%	13.3%
Add digit 4	5.1%	8.1%	8.0%	8.0%	8.0%	8.0%
Add digit 5	12.1%	10.8%	10.9%	10.8%	10.8%	10.8%
Add digit 6	8.3%	5.5%	5.4%	5.3%	5.3%	5.3%
Add digit 7	13.4%	13.5%	13.2%	13.3%	13.3%	13.3%
Add digit 8	4.9%	7.4%	8.0%	8.0%	8.0%	8.0%
Add digit 9	9.7%	14.2%	14.5%	14.6%	14.6%	14.6%

NOTE: odd digits appear more because can't have multiples of 4 the easiest square free killing entity

Walk to Infinity!

Theorem (Walking to Infinity on Square-Free Numbers)

Given we append one digit at a time (to the right), the probability that there is an infinite random square-free walk from any starting point is as least $1 - l \approx 0.99991$*. In other words, there is such a walk from almost any starting point.*

Proof comments:

- Pick a square-free number, denote by P_k the probability that longest walk with that starting point has length at most k
- Recursive formula: $P_{k+1} = (1 - p + pP_k)^{10}$
- See that $\lim_{k\to\infty} P_k =: \ell \approx 8.5950 \times 10^{-5}$

Important note: there do exist square-free numbers that are “dead ends," e.g. 231546210170694222

Concluding Remarks

Summary:

- The primes are a zero-density sequence while the square-free numbers have positive density
- Can't walk to infinity on primes if appending up to 2 digits a time in base 2, or in bases 3, 4, 5, or 6 when appending 1 digit at a time
- Can walk to infinity randomly on square-free numbers from almost any starting point

Questions?

Email: jsiktar@vols.utk.edu
ArXiV Link: https://arxiv.org/pdf/2010.14932.pdf