A Tale of Two Uniqueness Problems: Optimal Control in Solid Mechanics

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Outline

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What is Continuum Mechanics?

Definition (Continuum Mechanics)

Continuum mechanics is a classical differential equation model used to describe the interaction and movement of particles in a material

Features:

- Comprises both solid and fluid mechanics
- Assumes materials fill the entire body
- Same makeup if material is divided into pieces
- Adheres to Newton's Second law (resulting in a PDE)

What is Peridynamics?

Definition (Peridynamics)

Peridynamics (PD) is a nonlocal model for elasticity of solids that uses integrals over derivatives, attributed to Stewart A. Silling

Features:

- Exchanges derivatives in continuum models for integrals (helps address crack formation)
- Treats particles as having a bond between them
- Range of interaction parameterized by δ, called **horizon**
- Material parameters represented by $h(x)$ (e.g., density)
- Operator is elliptic (not parabolic or hyperbolic)

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Canonical Nonlocal Function Space: Fractional Sobolev Space

For $s \in (0,1)$, define the function space

$$W^{s,2}(\Omega) \;:=\; \left\{ u \in L^2(\Omega), \frac{|u(x)-u(y)|}{|x-y|^{\frac{n}{2}+s}} \in L^2(\Omega \times \Omega) \right\}$$

with associated norm

$$\|u\|_{W^{s,2}(\Omega)} \;:=\; \|u\|_{L^2(\Omega)} + \left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} dxdy \right)^{\frac{1}{2}} .$$

- Inspired as an intermediary between $L^2(\Omega)$ and $W^{1,2}(\Omega)$
- Theoretical properties inspire those for other nonlocal spaces (continuous embeddings, Hilbert space theory, etc.)

Formulation

Nonlocal equations [or systems] take the form

$$\begin{cases} L_\delta u = g, x \in \Omega \\ u = 0, \quad x \in \Omega_\delta \setminus \Omega \end{cases}$$

For our problem, k_δ is a non-negative, radial, integrable kernel, and

$$L_\delta u(x) = \frac{1}{2} \int_{\Omega_\delta} H(x,y) \frac{k_\delta(x-y)}{|x-y|^2} Du(x,y) dy$$

$$H(x,y) = \frac{h(x)+h(y)}{2}, 0 < h_{max} \leq h_\infty < \infty.$$

Projected difference (nonlocal linearized strain):

$$Du(x,y) := (u(x) - u(y)) \cdot \frac{x-y}{|x-y|}$$

Notation

Inner product:

$$B_h(u,v) \ := \ \int_{\Omega_\delta}\int_{\Omega_\delta} H(x,y)k_\delta(x-y)\frac{Du(x,y)Dv(x,y)}{|x-y|^2}dxdy$$

Function spaces:

$$X(\Omega_\delta;\mathbb{R}^n) := \{u \in L^2(\Omega_\delta;\mathbb{R}^n), B_h(u,u) < \infty\}$$

$$\partial X := \{w|_{\Omega_\delta \setminus \Omega}, w \in X\}$$

$$X_0(\Omega_\delta;\mathbb{R}^n) := \{u \in X, u = 0 \text{ in } \Omega_\delta \setminus \Omega\}$$

Existence-Uniqueness Theorem I

Theorem (Existence and Uniqueness)

For any $u_0 \in \partial X$ and $g \in L^2(\Omega;\mathbb{R}^n)$, $\exists! u \in u_0 + X_0$ such that the state system

$$B_h(u,w) \;=\; \int_\Omega g(x)\cdot w(x),\;\; w \in X_0.$$

is satisfied for all $w \in X_0$. Furthermore, we have the stability estimate

$$\|u\|_X \;\leq\; C(\|\widetilde{u}\|_X + \|g\|_{X*})$$

for some $C > 0$ independent of δ, where $\widetilde{u}$ is an extension of u_0 to all of Ω_δ.

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What is Optimal Control?

Definition

Optimal control is the study of minimizing cost functionals over classes of ordered pairs, where the coordinates are **controls** and **states**.

Features:

- Control and state are typically linked by solution maps
- Direct method often used to find existence of optimal controls
- Physical/biological context motivates a constraint
- Differential (or integral) equation constraints

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Formulation

Find $(\bar{u}, \bar{g}) \in (u_0 + X_0) \times L^2$ such that

$$I_\delta(\bar{u}, \bar{g}) \;=\; \min_{g \in L^2(\Omega;\mathbb{R}^n),\; u \in u_0 + X_0(\Omega_\delta;\mathbb{R}^n)} \left\{ \int_\Omega F(x, u(x))dx + \frac{\lambda}{2}\|g\|^2_{L^2(\Omega;\mathbb{R}^n)} \right\},$$

where u and g satisfy

$$B_h(u, w) \;=\; \int_\Omega g(x) \cdot w(x),\;\; w \in X_0.$$

Here g is an external force and u is displacement

Sample Cost Functionals

Example

Object Fitting:

$$I_\delta(u,g) \;:=\; \int_\Omega (u(x) - u_{des}(x))^2 + \frac{1}{2}\|g\|^2_{L^2(\Omega;\mathbb{R}^3)}$$

where u_{des} is the optimal shaping of a material in $\mathbb{R}^3$ to fit in a hole

Example

Work: $W = Fd$ from physics

$$I_\delta(u,g) \;:=\; \int_\Omega u(x)\cdot g(x)dx$$

Direct Method: Abstract Framework

Theorem

Let Z_{ad} be a nonempty, closed, bounded, and convex subset of Z. The Banach Space optimization problem

$$\min_{g \in Z_{ad}} \left\{ f(g) \; := \; G(Sg) + \frac{\lambda}{2} \|g\|_Z^p \right\}$$

has an optimal solution $\bar{g}$ if either of these conditions holds:

1. *$S : Z \to Y$ be is compact, and $G : Y \to \mathbb{R}$ is lower semi-continuous*
2. *$S : Z \to Y$ be is continuous and $G : Y \to \mathbb{R}$ is convex and lower semi-continuous*

Furthermore, if $\lambda > 0$, and G and S are linear on their respective domains (or G is convex), then there is a unique minimizer

Outline of Direct Method

- Show cost functional is bounded from below
- Pick a sequence of pairs approaching the infimum
- Use compactness properties to obtain suitable sub-sequence
- Show limit of sub-sequence satisfies constraint
- Uniqueness: contradiction/convexity argument

Carrying Out the Direct Method

$$I_\delta(\bar{u},\bar{g}) \;=\; \min_{g\in L^2(\Omega;\mathbb{R}^n),\; u\in u_0+X_0(\Omega;\mathbb{R}^n)} \left\{ \int_\Omega F(x,u(x))dx + \frac{\lambda}{2}\|g\|^2_{L^2(\Omega;\mathbb{R}^n)} \right\},$$

- Show main term is bounded from below assuming boundedness of Z_{ad} (oftentimes $Z_{ad} = \{z \in Z, a \le z \le b\}$)
- Find minimizing sequence
- Use compact embedding $X_0(\Omega_\delta;\mathbb{R}^n) \subset\subset L^2(\Omega_\delta;\mathbb{R}^n)$ to get convergence of sequences
- Show limit satisfies the state equation

Outline

What is Optimal Design?

Definition

Optimal design is an optimal control problem where a material is chosen to adhere to a specific force-displacement behavior as closely as possible

Prototypical design (scalar-valued):

$$\begin{cases} \min_{(h,u)\in\mathcal{H}\times X_0} \int_{\Omega_\delta}\int_{\Omega_\delta} F(x', x, u', u)dx'dx \\ L_\delta(u) = f(x) \text{ in } \Omega, \quad u = 0 \text{ in } \Omega_\delta \setminus \Omega \end{cases}$$

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Selected References

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