Analysis of Optimal Control and Optimal Design Problems in Peridynamics

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Find $(\overline{u_\delta}, \overline{g_\delta}) \in X_0 \times Z_{\text{ad}}$ such that

$$I(\overline{u_\delta}, \overline{g_\delta}) \; = \; \min_{u_\delta \in X_0, g_\delta \in Z_{\text{ad}}} \left\{ \int_\Omega F(x, u_\delta(x))dx + \frac{\lambda}{2} \int_\Omega \Gamma(x)|g_\delta(x)|^2 dx \right\}.$$

The minimization is over pairs $(g_\delta, u_\delta) \in Z_{\text{ad}} \times X_0(\Omega_\delta; \mathbb{R}^n)$ that satisfy

$$B_{\delta,\mathfrak{A}}(u_\delta, v) \; = \; \langle g_\delta, v \rangle, \text{ for all } \; v \in X_0(\Omega_\delta; \mathbb{R}^n).$$

Find $(\overline{\mathfrak{A}_\delta}, \overline{u_\delta}) \in \mathcal{H} \times X_0$ such that

$$J(\overline{\mathfrak{A}_\delta}, \overline{u_\delta}) \;=\; \min_{\mathfrak{A}_\delta \in \mathcal{H}, u_\delta \in X_0} \left\{ \int_\Omega F(x, u_\delta(x))dx + \iint_{\Omega\times\Omega} A(x, y, \mathfrak{A}_\delta(x, y))dxdy \right\}.$$

The minimization is over pairs $(\mathfrak{A}_\delta, u_\delta) \in \mathcal{H} \times X_0(\Omega_\delta; \mathbb{R}^n)$ that satisfy

$$B_{\delta,\mathfrak{A}_\delta}(u_\delta, v) \;=\; \langle g, v\rangle, \text{ for all } \; v \in X_0(\Omega_\delta; \mathbb{R}^n).$$

Outline

1. Overarching Ideas
2. Notation
3. Optimal Control: Analysis
4. Optimal Control: Discretization
5. Optimal Design: Analysis
6. Closing Remarks

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1. **Overarching Ideas**
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What is peridynamics?

Definition (PD)

Peridynamics (PD) is a nonlocal model for elasticity of solids that uses integrals over derivatives, attributed to Stewart A. Silling

Features:

- Exchanges derivatives in continuum models for integrals (helps address crack formation), making model inherently **nonlocal**
- Treats particles as having a bond between them (**bond-based model**)
- Range of interaction parameterized by δ, called **horizon**
- Material parameters represented by $\mathfrak{a}(x)$ (e.g., density)
- Operator is elliptic (not parabolic or hyperbolic)

Sample Candidate Integrand

Example:

$$F(x, u(x)) = |u(x) - u_{\text{des}}(x)|^2$$

where u_{des} is the optimal shape of the material in space to fit a pre-determined hole as closely as possible

Optimal control versus optimal design

Similarities:

- Involve a minimization of a [cost] functional over pairs in a product function space
- Utilize tools of calculus of variations for abstract well-posedness
- Can be meaningfully cast in non-local settings
- Forces and displacements are vector-valued

Differences:

- Optimal control: material is fixed but force may vary
- Optimal design: force is fixed but material will vary
- Existence of candidate minimizers are proven using different tools (more on this later!)

Outline

1 Overarching Ideas

2 **Notation**

3 Optimal Control: Analysis

4 Optimal Control: Discretization

5 Optimal Design: Analysis

6 Closing Remarks

Notation

- Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $\Omega_\delta := \Omega \cup \{x, \mathrm{dist}(x, \partial\Omega) < \delta\}$
- $\Omega_\delta \setminus \Omega$ is non-local boundary
- $\mathcal{D}_\delta := (\Omega \times \Omega_\delta) \cup (\Omega_\delta \times \Omega)$
- Projected difference: $Du(x, y) := \frac{(u(x)-u(y))\cdot(x-y)}{|x-y|}$, nonlocal linearized strain (for vector-valued functions)
- Our class of design coefficients, denoted by $\mathcal{H}$, is all coefficients of the form

$$\mathfrak{A}(x, y) \;:=\; \frac{\mathfrak{a}(x) + \mathfrak{a}(y)}{2},$$

where there exist $a_{\min}, a_{\max} > 0$ so $a_{\min} \leq \mathfrak{a} \leq a_{\max}$ on Ω_δ.

Assumptions on Kernels

Kernel sequence $\{k_\delta\}_{\delta>0}$ radial, integrable, non-negative, supported in $B(0,\delta)$, $k_\delta(r)r^{-2}$ is nonincreasing, and

$$\lim_{\delta\to 0^+}\int_{\mathbb{R}^n} k_\delta(\xi)d\xi \;=\; \delta_0$$

Also, for all $\delta > 0$:

$$\int_{\mathbb{R}^n} k_\delta(\xi)d\xi \;=\; 1$$

$$\lim_{t\to\infty}\int_{\mathbb{R}^n\setminus B(0,t)} k_\delta(\xi)d\xi \;=\; 0$$

Bi-linear forms

Nonlocal bi-linear form:

$$B_{\delta,\mathfrak{A}}(u,v) \;:=\; \frac{1}{2}\iint_{\mathcal{D}_\delta} \mathfrak{A}(x,y)k_\delta(x-y)\frac{Du(x,y)}{|x-y|}\frac{Dv(x,y)}{|x-y|}dxdy$$

Local bi-linear form:

$$B_{0,\mathfrak{A}}(u,v) \;:=\; C(n)\int_\Omega \mathfrak{a}(x)(2\langle \mathrm{Sym}(\bigtriangledown u), \mathrm{Sym}(\bigtriangledown v)\rangle_F + \mathrm{div}(u)\mathrm{div}(v))dx,$$

with $C(n) = \frac{1}{(n+2)(n+4)}$

Inner products denoted $\langle\cdot,\cdot\rangle_Y$; L^2-inner product denoted $\langle\cdot,\cdot\rangle$

Function Spaces

Our function space is based on $B_{\delta,\mathfrak{A}}$:

$$X(\Omega_\delta;\mathbb{R}^n) := \{u|_\Omega \in L^2(\Omega;\mathbb{R}^n), B_{\delta,\mathfrak{A}}(u,u) < \infty\}$$

Version with zero non-local boundary data:

$$X_0(\Omega_\delta;\mathbb{R}^n) := \{u \in X(\Omega_\delta;\mathbb{R}^n), u = 0 \text{ in } \Omega_\delta \setminus \Omega\}$$

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Structural Properties

The following hold:

- The space $C_0^\infty(\Omega;\mathbb{R}^n)$ is dense in $X_0(\Omega_\delta;\mathbb{R}^n)$
- Functions in $X_0(\Omega_\delta;\mathbb{R}^n)$ can be extended by zero to obtain functions in $X(\mathbb{R}^n;\mathbb{R}^n)$
- Compact embedding: $X(\Omega_\delta;\mathbb{R}^n) \Subset L^2(\Omega;\mathbb{R}^n)$

NOTE: The first two statements become open questions if we remove the boundary data

Preliminaries (continued)

Proposition (Nonlocal Poincaré-Korn (Mengesha-Du 2014))

There exists a $\delta_0 > 0$ *and a constant* $C(\delta_0) > 0$ *such that for all* $\delta \in (0, \delta_0]$ *and* $u \in X_0(\Omega_\delta; \mathbb{R}^n)$,

$$\|u\|^2_{L^2(\Omega;\mathbb{R}^n)} \leq \; C(\delta_0) \int_{\Omega_\delta} \int_{\Omega_\delta} k_\delta(x-y) \frac{|Du(x,y)|^2}{|x-y|^2} dxdy.$$

NOTE: Independence of constant from δ will be important when we send $\delta \to 0^+$

State equation is well-posed!

Theorem (Existence and Uniqueness for State Equation)

For any $g_\delta \in L^2(\Omega;\mathbb{R}^n)$, there exists a unique $u_\delta \in X_0(\Omega_\delta;\mathbb{R}^n)$ such that the state system

$$B_{\delta,\mathfrak{A}}(u_\delta, w_\delta) \; = \; \langle g_\delta, w_\delta \rangle$$

is satisfied for all $w_\delta \in X_0$. Furthermore, we have the stability estimate

$$\|u_\delta\|_{X(\Omega_\delta;\mathbb{R}^n)} \lesssim \; \|g_\delta\|_{X(\Omega_\delta;\mathbb{R}^n)^*}$$

for some constant independent of δ.

Minimization Problem

Goal: find $(\overline{u_\delta}, \overline{g_\delta}) \in X_0 \times L^2$ minimizing

$$I(u_\delta, g_\delta) = \int_\Omega F(x, u_\delta(x))dx + \frac{\lambda}{2}\int_\Omega \Gamma(x)|g_\delta(x)|^2 dx$$

subject to: $\lambda \geq 0$, $g_\delta \in Z_{ad} \subset L^2$ and $(u_\delta, g_\delta) \in X_0 \times L^2$ solving

$$B_\delta(u_\delta, v_\delta) = \int_\Omega g_\delta(x) \cdot v_\delta(x)dx \quad \forall v_\delta \in X_0$$

Cost Functional Assumptions

$$I(u,g) \;:=\; \int_\Omega F(x,u(x))dx + \frac{\lambda}{2}\int_\Omega \Gamma(x)|g(x)|^2 dx$$

Here Z_{ad} is a nonempty, closed, convex, and bounded subset of $L^2(\Omega;\mathbb{R}^n)$, taking the form

$$Z_{\text{ad}} = \{z \in L^2(\Omega;\mathbb{R}^n), a \preceq z \preceq b\}$$

Here $[a]_i \leq [b]_i$ for all $i \in \{1,2,\ldots,n\}$ with $a = ([a]_1,\ldots,[a]_n)$ and $b = ([b]_1,\ldots,[b]_n)$ being vector fields in $L^2(\Omega;\mathbb{R}^n)$, $\lambda \geq 0$, and $\Gamma \in L^\infty(\Omega)$ is positive.

Cost Functional Assumptions (continued)

The integrand $F : \Omega \times \mathbb{R}^n \to \mathbb{R}$ possesses the following properties:

1. For all $v \in \mathbb{R}^n$ the mapping $x \mapsto F(x, v)$ is measurable;
2. For all $x \in \Omega$ the mapping $v \mapsto F(x, v)$ is continuous and convex;
3. There exist $c_1 > 0$ and $\ell \in L^1(\Omega)$

$$|F(x, v)| \leq c_1 |v|^2 + \ell(x)$$

for all $x \in \Omega$, $v \in \mathbb{R}^n$.

Well-posedness of optimal control problem

Theorem (Well-posedness)

There exists $(\overline{u_\delta}, \overline{g_\delta}) \in X_0(\Omega_\delta; \mathbb{R}^n) \times Z_{ad}$ *minimizing*

$$I(u_\delta, g_\delta) = \int_\Omega F(x, u_\delta(x))dx + \frac{\lambda}{2}\int_\Omega \Gamma(x)|g_\delta(x)|^2 dx,$$

where $\overline{u_\delta} \in X_0$ *solves*

$$B_{\delta,\mathfrak{A}}(u_\delta, v_\delta) = \int_\Omega g_\delta(x) \cdot v_\delta(x)dx \quad \forall v_\delta \in X_0$$

Furthermore, if F *is strictly convex or* $\lambda > 0$*, then the minimizer is unique.*

Use compactness to apply direct method

Optimality conditions

Let S_δ denote solution operator for state equation

Non-local continuous optimality conditions (including adjoint):

$$\begin{aligned} \overline{u_\delta} &= S_\delta \overline{g_\delta} \\ \overline{p_\delta} = S_\delta^* F_u(\cdot, \overline{u_\delta}) &= S_\delta F_u(\cdot, \overline{u_\delta}) \\ \langle \overline{p_\delta} + \lambda \overline{g_\delta}, \gamma_z - \overline{g_\delta} \rangle_{L^2} &\geq 0, \quad \forall \gamma_z \in Z_{\mathrm{ad}}. \end{aligned}$$

Projection formula:

$$\overline{g_\delta}(x) = \mathbb{P}_{Z_{\mathrm{ad}}} \left(-\frac{1}{\lambda} \overline{p_\delta}(x) \right)$$

NOTE: No second-order optimality conditions needed (strict convexity!)

Convergence of state equation as $\delta \to 0^+$

Local bi-linear form (of Navier-Lamé system of linear elasticity)

$$B_{0,\mathfrak{A}}(u,v) \;:=\; C(n)\int_\Omega \mathfrak{a}(x)(2\langle \mathrm{Sym}(\bigtriangledown u), \mathrm{Sym}(\bigtriangledown v)\rangle_F + \mathrm{div}(u)\mathrm{div}(v))dx$$

where $C(n) := \frac{1}{(n+2)(n+4)}$.

Theorem

Suppose $\{(\overline{u_\delta}, \overline{g_\delta})\}_{\delta>0}$ is the family of solutions to the non-local problem. Then, there is $(\overline{u}, \overline{g})$ such that $\overline{u_\delta} \to \overline{u}$ in $L^2(\Omega;\mathbb{R}^n)$ and $\overline{g_\delta} \to \overline{g}$ in $L^2(\Omega;\mathbb{R}^n)$. Moreover, $(\overline{u}, \overline{g})$ solves the local optimal control problem.

Local state equation that $(\overline{u}, \overline{g})$ satisfies the constraint:

$$B_{0,\mathfrak{A}}(u,v) \;=\; \int_\Omega g(x)\cdot v(x)dx$$

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Nonlocal discrete problem statement

Find $(\overline{u_{\delta,h}}, \overline{g_{\delta,h}}) \in X_{\delta,h} \times Z_h$ such that

$$I(\overline{u_{\delta,h}}, \overline{g_{\delta,h}}) = \min_{u_{\delta,h} \in X_{\delta,h},\ g_{\delta,h} \in Z_h} I(u_{\delta,h},\ g_{\delta,h}),$$

over pairs $(u_{\delta,h}, g_{\delta,h}) \in X_{\delta,h} \times Z_h$ that satisfy

$$B_{\delta,\mathfrak{A}}(u_{\delta,h}, v_{\delta,h}) \ = \ \langle g_{\delta,h}, v_{\delta,h} \rangle, \qquad \forall v_{\delta,h} \in X_{\delta,h}.$$

Recap:

$$I(u_{\delta,h}, g_{\delta,h}) \ := \ \int_\Omega F(x, u_{\delta,h}(x))dx + \frac{\lambda}{2} \|g_{\delta,h}\|^2_{L^2(\Omega;\mathbb{R}^n)}$$

Henceforth assume $\Gamma \equiv 1$

Local discrete problem statement

Find $(\overline{u_h}, \overline{g_h}) \in X_h \times Z_h$ such that

$$I(\overline{u_h}, \overline{g_h}) \;=\; \min_{u_h \in X_h,\; g_h \in Z_h} I(u_h, g_h),$$

over pairs $(u_h, g_h) \in X_h \times Z_h$ that satisfy

$$B_{0,\mathfrak{A}}(u_h, v_h) \;=\; \langle g_h, v_h \rangle, \qquad \forall v_h \in Z_h.$$

Finite element method notation

- Mesh family: $\{\mathscr{T}_h\}_{h>0}$ (discretizing Ω_δ) shape-regular and quasi-uniform
- Piecewise polynomials of degree m (with respect to our mesh):

$$\mathcal{P}_m(T;\mathbb{R}^n) := \left\{ \sum_{\alpha\in\mathbb{N}_0^n \,:\, \sum_{i=1}^n \alpha_i \le m} v_\alpha x_1^{\alpha_1}\cdots x_n^{\alpha_n} \;\middle|\; v_\alpha \in \mathbb{R}^n,\ (x_i)_{i=1}^n \in T \right\}$$

- Discretized state space: $X_{\delta,h} := X_h := \{w_h \in C^0(\overline{\Omega_\delta};\mathbb{R}^n) \mid w_h|_T \in \mathcal{P}_1(T;\mathbb{R}^n)\ \forall T \in \mathscr{T}_h, w_h = 0 \text{ on } \Omega_\delta \setminus \Omega\}$
- Discretized control space: $Z_h := \{z_h|_T \in \mathcal{P}_0(T;\mathbb{R}^n)\ \forall T \in \mathscr{T}_h\}$
- $\Pi_0 : Z_{\text{ad}} \to Z_h$ is piecewise constant projection by averages on each triangle

Optimality Conditions (discretized)

Non-local discrete optimality conditions (including adjoint)

$$\begin{aligned} \langle \overline{p_{\delta,h}} + \lambda \overline{g_{\delta,h}}, \gamma_h - \overline{g_{\delta,h}} \rangle &\geq 0, \quad \forall \gamma_h \in Z_{\text{ad}} \cap Z_h \\ \overline{p_{\delta,h}} = S^*_{\delta,h} F_u(\cdot, \overline{u_{\delta,h}}) &= S_{\delta,h} F_u(\cdot, \overline{u_{\delta,h}}) \\ \overline{u_{\delta,h}} &= S_{\delta,h} \overline{g_{\delta,h}}. \end{aligned}$$

Projection formula:

$$\overline{g_{\delta,h}}(x) = \mathbb{P}_{Z_{\text{ad}}} \left(-\frac{1}{\lambda} \Pi_0 \overline{p_{\delta,h}}(x) \right)$$

Intermediary Functions

By Lax-Milgram, we may define $\widehat{u_\delta}, \widehat{p_\delta} \in X_0(\Omega_\delta; \mathbb{R}^n)$ such that

$$B_{\delta,\mathfrak{A}}(\widehat{u_\delta}, v_\delta) \;=\; \langle \overline{g_{\delta,h}}, v_\delta \rangle \quad \forall v_\delta \in X_0(\Omega_\delta; \mathbb{R}^n);$$

$$B_{\delta,\mathfrak{A}}(v_\delta, \widehat{p_\delta}) \;=\; \langle v_\delta, \overline{u_{\delta,h}} \rangle \quad \forall v_\delta \in X_0(\Omega_\delta; \mathbb{R}^n);$$

also define $\widehat{u_h}, \widehat{p_h} \in H_0^1(\Omega; \mathbb{R}^n)$ such that

$$B_{0,\mathfrak{A}}(\widehat{u_h}, v) \;=\; \langle \overline{g_h}, v \rangle \quad \forall v \in H_0^1(\Omega; \mathbb{R}^n);$$

$$B_{0,\mathfrak{A}}(v, \widehat{p_h}) \;=\; \langle v, \widehat{u_h} \rangle \quad \forall v \in H_0^1(\Omega; \mathbb{R}^n).$$

State and Adjoint Error Estimates

Theorem (State and Adjoint Error Estimates)

Suppose that $(\overline{u_{\delta,h}}, \overline{g_{\delta,h}})$ *is the solution to the non-local discrete problem,* $\overline{p_{\delta,h}}$ *solves the discrete adjoint equation given* $\overline{u_{\delta,h}}$*;* $(\overline{u_{\delta}}, \overline{g_{\delta}})$ *is the solution to the nonlocal continuous problem; and* $\overline{p_{\delta}}$ *solves the continuous adjoint equation given* $\overline{u_{\delta}}$*. Then*

$$\|\overline{u_\delta} - \overline{u_{\delta,h}}\|_{X(\Omega_\delta;\mathbb{R}^n)} \lesssim \inf_{v_{\delta,h}\in X_{\delta,h}} \|\widehat{u_\delta} - v_{\delta,h}\|_{X(\Omega_\delta;\mathbb{R}^n)} + \|\overline{g_\delta} - \overline{g_{\delta,h}}\|_{L^2(\Omega;\mathbb{R}^n)};$$

$$\|\overline{p_\delta} - \overline{p_{\delta,h}}\|_{X(\Omega_\delta;\mathbb{R}^n)} \lesssim \inf_{v_{\delta,h}\in X_{\delta,h}} \|\widehat{p_\delta} - v_{\delta,h}\|_{X(\Omega_\delta;\mathbb{R}^n)} + \inf_{v_{\delta,h}\in X_{\delta,h}} \|\widehat{u_\delta} - v_{\delta,h}\|_{X(\Omega_\delta;\mathbb{R}^n)} + \|\overline{g_\delta} - \overline{g_{\delta,h}}\|_{L^2(\Omega;\mathbb{R}^n)}.$$

Auxiliary lemmas

Lemma (Regularity of Control for Fractional-Type Kernels)

Suppose that

$$\frac{k_\delta(\xi)}{|\xi|^2} \sim \frac{1}{|\xi|^{n+2s}}$$

holds for all $\xi \in B(0,\delta)$, *for some* $s \neq \frac{1}{2}$. *Then necessarily* $\overline{g_\delta} \in X(\Omega_\delta; \mathbb{R}^n)$.

Recall Fractional Sobolev Space (special case of our kernels):

$$H^s(\Omega_\delta; \mathbb{R}^n) \ := \ \left\{ u|_\Omega \in L^2(\Omega; \mathbb{R}^n), \frac{|u(x) - u(y)|}{|x-y|^{\frac{n}{2}+s}} \in L^2(\Omega_\delta \times \Omega_\delta) \right\}$$

Proof strategy:

- Mengesha-Du 2016 says that $H^s(\Omega_\delta; \mathbb{R}^n) = X(\Omega_\delta; \mathbb{R}^n)$ here
- Use projection formula $\overline{g_\delta}(x) = -\frac{1}{\lambda}\mathbb{P}_{Z_{\text{ad}}}(\overline{p_\delta}(x))$
- This pointwise projection is continuous in H^s semi-norm but not X semi-norm!

Control Convergence

Theorem (Convergence of Controls)

Assume that $\overline{g_\delta}$ is the optimal control associated with the nonlocal continuous problem, and $\overline{g_{\delta,h}}$ be the discrete optimal control. Then we have the convergence

$$\|\overline{g_\delta} - \overline{g_{\delta,h}}\|_{L^2(\Omega;\mathbb{R}^n)} \lesssim \omega(h) + \inf_{v_{\delta,h}\in X_{\delta,h}} [\overline{u_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)} + \inf_{v_{\delta,h}\in X_{\delta,h}} [\overline{p_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)}.$$

Non-local Problem Convergence: Summary

Corollary (Full Norm Solution Convergence)

In the setting of our problem formulation,

$$\|\overline{u_\delta} - \overline{u_{\delta,h}}\|_{X(\Omega_\delta;\mathbb{R}^n)} \lesssim \omega(h) + \inf_{v_{\delta,h}\in X_{\delta,h}} \|\widehat{u_\delta} - v_{\delta,h}\|_{X(\Omega_\delta;\mathbb{R}^n)} + \inf_{v_{\delta,h}\in X_{\delta,h}} [\overline{u_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)} + \inf_{v_{\delta,h}\in X_{\delta,h}} [\overline{p_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)};$$

$$\|\overline{p_\delta} - \overline{p_{\delta,h}}\|_{X(\Omega_\delta;\mathbb{R}^n)} \lesssim \omega(h) + \inf_{v_{\delta,h}\in X_{\delta,h}} \|\widehat{p}_h - v_{\delta,h}\|_{X(\Omega_\delta;\mathbb{R}^n)} + \inf_{v_{\delta,h}\in X_{\delta,h}} \|\widehat{u}_\delta - v_{\delta,h}\|_{X(\Omega_\delta;\mathbb{R}^n)} + \inf_{v_{\delta,h}\in X_{\delta,h}} [\overline{u_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)} + \inf_{v_{\delta,h}\in X_{\delta,h}} [\overline{p_\delta} - v_{\delta,h}]_{X(\Omega_\delta;\mathbb{R}^n)}.$$

Local Convergence Results

Theorem

Suppose $(\overline{u}, \overline{g})$ denotes the solution to the local continuous problem, while $(\overline{u_h}, \overline{g_h})$ is the solution to the local discrete problem, and $\overline{p}$ and $\overline{p_h}$ solve the respective adjoint problems. We have the estimates

$$\|\overline{u} - \overline{u_h}\|_{H^1(\Omega;\mathbb{R}^n)} \lesssim \inf_{v_h \in X_h} [\widehat{u} - v_h]_{H^1(\Omega;\mathbb{R}^n)} + \|\overline{g} - \overline{g_h}\|_{L^2(\Omega;\mathbb{R}^n)};$$

$$\|\overline{p} - \overline{p}\|_{H^1(\Omega;\mathbb{R}^n)} \lesssim \inf_{v_h \in X_h} [\widehat{p} - v_h]_{H^1(\Omega;\mathbb{R}^n)} + \inf_{v_h \in X_h} [\widehat{u} - v_h]_{H^1(\Omega;\mathbb{R}^n)} + \|\overline{g} - \overline{g_h}\|_{L^2(\Omega;\mathbb{R}^n)}.$$

$$\|\overline{g} - \overline{g_h}\|_{L^2(\Omega;\mathbb{R}^n)} \lesssim h + \inf_{v_h \in X_h} [\overline{p} - v_h]_{H^1(\Omega;\mathbb{R}^n)} + \inf_{v_h \in X_h} [\overline{u} - v_h]_{H^1(\Omega;\mathbb{R}^n)}.$$

Discrete Analogues of Convergence

Theorem (Discrete Convergence)

Suppose $\{(\overline{u_{\delta,h}}, \overline{g_{\delta,h}})\}_{\delta>0}$ is the family of solutions to the non-local discrete problem. Then, there is $(\overline{u_h}, \overline{g_h})$ such that $\overline{u_{\delta,h}} \to \overline{u_h}$ in $L^2(\Omega;\mathbb{R}^n)$ and $\overline{g_{\delta,h}} \to \overline{g_h}$ in $L^2(\Omega;\mathbb{R}^n)$. Moreover, $(\overline{u_h}, \overline{g_h})$ solves the local discrete optimal control problem.

What is asymptotic compatibility?

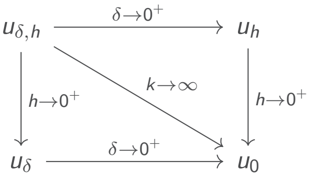
- Introduced by X. Tian and Q. Du (2014)
- Originally developed for linear, non-local state equations

$$\mathcal{L}_{\delta,h} u_{\delta,h} = f$$

- Unconditional convergence of approximations in both discretization and horizon parameters

Definition (Asymptotic Compatibility)

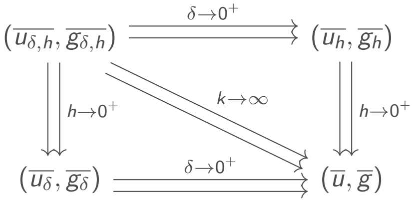
Given fixed data f in a Hilbert Space, the family of solutions $\{u_{\delta,h}\}_{\delta,h>0}$ is **asymptotically compatible** in $\delta, h > 0$ if for any sequences $\{\delta_k\}_{k=1}^{\infty}, \{h_k\}_{k=1}^{\infty}$ with $\delta_k, h_k \to 0$, we have that $u_{\delta_k,h_k} \to u_0$ strongly in some Hilbert space norm, where u_0 is the solution to a local, continuous problem.



What is asymptotic compatibility? (continued)

Definition (Asymptotic Compatibility for Optimal Control)

We say that the family of solutions $\{(\overline{u_{\delta,h}}, \overline{g_{\delta,h}})\}_{h>0,\delta>0}$ to the nonlocal discrete optimal control problem is **asymptotically compatible** in $\delta, h > 0$ if for any sequences $\{\delta_k\}_{k=1}^{\infty}, \{h_k\}_{k=1}^{\infty}$ with $\delta_k, h_k \to 0$, we have that $\overline{g_{\delta_k,h_k}} \to \overline{g}$ strongly in $L^2(\Omega;\mathbb{R}^n)$, and $\overline{u_{\delta_k,h_k}} \to \overline{u}$ strongly in $L^2(\Omega;\mathbb{R}^n)$. Here $(\overline{u}, \overline{g}) \in H_0^1(\Omega;\mathbb{R}^n) \times Z_{\text{ad}}$ denotes the optimal solution for the local continuous problem.



Theorem

Assume $\mathfrak{A}$ is Lipschitz. Then, our family of optimal control problems is asymptotically compatible as $\delta, h \to 0^+$.

Asymptotic compatibility: result and notation

Define $A_\delta : X_0(\Omega_\delta;\mathbb{R}^n) \to (X_0(\Omega_\delta;\mathbb{R}^n))^*$ as the invertible operator such that

$$\langle A_\delta u, v\rangle_{X_0^*,X_0} \;=\; B_{\delta,\mathfrak{A}}(u,v) \quad \forall u,v \in X_0(\Omega_\delta;\mathbb{R}^n);$$

define $A_0 : H_0^1(\Omega;\mathbb{R}^n) \to H^{-1}(\Omega;\mathbb{R}^n)$ as the invertible operator such that

$$\langle A_0 u, v\rangle_{H^{-1},H_0^1} \;=\; B_{0,\mathfrak{A}}(u,v) \quad \forall u,v \in H_0^1(\Omega;\mathbb{R}^n).$$

Proof of asymptotic compatibility

Needed preliminaries:

1. Given a $v \in H^1_0(\Omega;\mathbb{R}^n)$, and $h_k, \delta_k \to 0$, we can find a sequence $v_k \in X_{\delta_k,h_k}$ such that $v_k \to v$ strongly in $H^1(\Omega;\mathbb{R}^n)$ as $k \to \infty$.
2. For any sequences $\{\delta_k\}_{k=1}^{\infty}, \{h_k\}_{k=1}^{\infty}$ with $\delta_k, h_k \to 0$, there exists a $C > 0$ so that $\|\overline{u_{\delta_k,h_k}}\|_{X(\Omega_{\delta_k};\mathbb{R}^n)} \leq C$ uniformly in $k \in \mathbb{N}^+$.
3. We have that $A_\delta u \in L^2(\Omega;\mathbb{R}^n)$ and $\lim_{\delta\to 0^+}\|A_\delta u - A_0 u\|_{L^2(\Omega;\mathbb{R}^n)} = 0$ for each $u \in C_0^\infty(\Omega;\mathbb{R}^n)$.

Pick sub-sequence of $\{(\overline{u_k}, \overline{g_k}, \overline{p_k})\}_{k=1}^{\infty}$ so there is a limit point $(u_*, g_*, p_*) \in H^1_0(\Omega;\mathbb{R}^n) \times Z_{\text{ad}} \times H^1_0(\Omega;\mathbb{R}^n)$ with convergence in appropriate topology (weak convergence of controls)

Proof of asymptotic compatibility (continued)

Step 1: Show that $B_{0,\mathfrak{A}}(u_*, \varphi) \;=\; \langle g_*, \varphi\rangle$ for all $\varphi \in H_0^1(\Omega;\mathbb{R}^n)$
Pick $\varphi \in C_0^\infty(\Omega;\mathbb{R}^n)$, let $w_k := I_k\varphi$ (nodal interpolation), $w_k \to \varphi$ in $W^{1,\infty}(\Omega;\mathbb{R}^n)$, then compute limit as $k \to \infty$ of

$$B_{\delta_k,\mathfrak{A}}(\overline{u_k}, w_k) \;=\; \langle A_{\delta_k}\varphi, \overline{u_k}\rangle_{X_0^*,X_0} + \langle A_{\delta_k}(w_k - \varphi), \overline{u_k}\rangle_{X_0^*,X_0} \;=:\; I_k + II_k.$$

Step 2: Show that $B_{0,\mathfrak{A}}(\varphi, p_*) \;=\; \langle u_*, \varphi\rangle$ for all $\varphi \in H_0^1(\Omega;\mathbb{R}^n)$
Strategy is identical to Step 1!

Step 3: Show that $g_*(x) \;=\; \mathbb{P}_{Z_{\text{ad}}}\left(-\frac{1}{\lambda}p_*(x)\right)$
Recall that $\overline{g_k}(x) \;=\; \mathbb{P}_{Z_{\text{ad}}}\left(-\frac{1}{\lambda}\Pi_0\overline{p_k}(x)\right)$, show $\Pi_0\overline{p_k} \to p_*$ strongly in $L^2(\Omega;\mathbb{R}^n)$

Proof of asymptotic compatibility (continued)

Step 4: Unraveling

- Steps 1-3 and uniqueness of solutions to optimality system give $\overline{u} = u_*$, $\overline{g} = g_*$, and $\overline{p} = p_*$
- This is the limit point reached for any sub-sequence of original sequence
- Entire sequence of triples $\{(\overline{u_k}, \overline{g_k}, \overline{p_k})\}_{k=1}^{\infty}$ converges to $(\overline{u}, \overline{g}, \overline{p})$

Step 5: Strong convergence of controls
Use the Lipschitz property of the projection and the estimate

$$\|\overline{g_k} - \overline{g}\|_{L^2(\Omega;\mathbb{R}^n)} \lesssim \|\overline{p} - \Pi_0\overline{p}\|_{L^2(\Omega;\mathbb{R}^n)} + \|\Pi_0\overline{p} - \Pi_0\overline{p_k}\|_{L^2(\Omega;\mathbb{R}^n)}$$

Outline

1. Overarching Ideas
2. Notation
3. Optimal Control: Analysis
4. Optimal Control: Discretization
5. **Optimal Design: Analysis**
6. Closing Remarks

Homogenization general setup

If we have the family of problems

$$\begin{cases} -\text{div}(A_j \bigtriangledown u_j) \; = \; f, \; \text{in } \Omega \\ u_j \; = \; 0 \qquad\qquad\qquad \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is bounded, $f \in H^{-1}(\Omega)$, and $\{A_j\}_{j=1}^{\infty}$ are symmetric and uniformly coercive matrices, we want to find a homogenized limit, i.e. a pair (A, u) solving

$$\begin{cases} -\text{div}(A \bigtriangledown u) \; = \; f, \; \text{in } \Omega \\ u \; = \; 0 \qquad\qquad\qquad \text{on } \partial\Omega \end{cases}$$

More notation

Nonlocal gradient:

$$D_{k_\delta} u(x,y) \ := \ k_\delta(x-y)^{\frac{1}{2}} \frac{Du(x,y)}{|x-y|},$$

Nonlocal flux (for $\mathfrak{A} \in \mathcal{H}$):

$$\Psi_{\delta,\mathfrak{A}} u(x,y) \ := \ \mathfrak{A}(x,y) k_\delta(x-y)^{\frac{1}{2}} \frac{Du(x,y)}{|x-y|},$$

NOTE: This is a nonlocal analogue of the quantity $-\mathrm{div}(A \bigtriangledown u)$ in homogenization problems!

Nonlocal divergence operator:

$$d_{k_\delta} \phi(x) \ := \ PV \int_{\Omega_\delta} \frac{k_\delta(x-y)^{\frac{1}{2}}}{|x-y|} (\phi(x,y) + \phi(y,x)) \frac{x-y}{|x-y|} dy$$

Structural Lemmas

Lemma

If $\{\mathfrak{A}_j\}_{j=1}^{\infty} \subset \mathcal{H}$ *is such that* $\mathfrak{A}_j \overset{*}{\rightharpoonup} \mathfrak{A}$ *weak-** $L^\infty(\Omega_\delta \times \Omega_\delta)$, *then* $\mathfrak{A} \in \mathcal{H}$.

Lemma (Nonlocal Integration by Parts)

If $\phi \in L^2(\Omega_\delta \times \Omega_\delta)$, *then* $d_{k_\delta}\phi$ *belongs to* $X(\Omega_\delta; \mathbb{R}^n)^*$. *In addition, if* $u \in X(\Omega_\delta; \mathbb{R}^n)$, *we have the following integration by parts formula:*

$$\int_{\Omega_\delta}\int_{\Omega_\delta} \phi(x,y) D_{k_\delta} u(x,y) dx dy \;=\; \langle d_{k_\delta}\phi, u\rangle.$$

Well-posedness of state equation

Proposition

For any $\mathfrak{A} \in \mathcal{H}$*, the state equation has a unique solution* $u_\delta \in X_0(\Omega_\delta; \mathbb{R}^n)$*, characterized by the minimization* $u_\delta := \text{argmin}_{v \in X_0(\Omega_\delta;\mathbb{R}^n)} \mathscr{J}^\delta_{\mathfrak{A}}(v)$*, where*

$$\mathscr{J}^\delta_{\mathfrak{A}}(u) \ := \ \frac{1}{2} \iint_{\mathcal{D}_\delta} \mathfrak{A}(x,y) k_\delta(|x-y|) \left| \frac{Du(x,y)}{|x-y|} \right|^2 dxdy - \int_\Omega g(x) \cdot u(x) dx$$

Furthermore, the minimizer satisfies the estimate

$$\|u_\delta\|_{X(\Omega_\delta;\mathbb{R}^n)} \lesssim \ \|g\|_{L^2(\Omega_\delta;\mathbb{R}^n)},$$

where the implicit constant is independent of δ*.*

Nonlocal G-convergence

Theorem (Nonlocal G-convergence)

If $\{\mathfrak{A}_{\delta,j}\}_{j=1}^{\infty} \subset \mathcal{H}$ is such that $\mathfrak{A}_{\delta,j} \overset{*}{\rightharpoonup} \mathfrak{A}_{\delta}$ in the weak- $L^{\infty}(\Omega_{\delta} \times \Omega_{\delta})$ topology, then $u_{\delta,j} \rightharpoonup u_{\delta}$ weakly in $X(\Omega_{\delta}; \mathbb{R}^n)$, where $u_{\delta,j} := \mathcal{L}^{-1}_{\delta,\mathfrak{A}_{\delta,j}} g$ and $u_{\delta} := \mathcal{L}^{-1}_{\delta,\mathfrak{A}_{\delta}} g$.*

- Use boundedness of sequence $\{u_{\delta,j}\}_{j=1}^{\infty}$ to produce a weak limit $\widetilde{u_{\delta}}$
- Use weak-* convergence to prove $\lim_{j\to\infty} \mathscr{J}^{\delta}_{\mathfrak{A}_{\delta,j}}(u_{\delta,j}) \leq \mathscr{J}^{\delta}_{\mathfrak{A}_{\delta}}(\widetilde{u_{\delta}})$.
- Use Generalized Fatou's Lemma to prove reverse inequality
- Use uniqueness of minimizers of $\mathscr{J}^{\delta}_{\mathfrak{A}_{\delta}}$ to conclude

NOTE: Such a result does not hold in the local setting!

Nonlocal H-convergence

Theorem (Nonlocal H-convergence)

If $\{\mathfrak{A}_{\delta,j}\}_{j=1}^{\infty} \subset \mathcal{H}$ is such that $\mathfrak{A}_{\delta,j} \overset{*}{\rightharpoonup} \mathfrak{A}_\delta$ in the weak- $L^\infty(\Omega_\delta \times \Omega_\delta)$ topology, then $u_{\delta,j} \rightharpoonup u_\delta$ weakly in $X(\Omega_\delta;\mathbb{R}^n)$, where $u_{\delta,j} := \mathcal{L}^{-1}_{\delta,\mathfrak{A}_{\delta,j}} g$ and $u_\delta := \mathcal{L}^{-1}_{\delta,\mathfrak{A}_\delta} g$. In addition, we have $\Psi_{\delta,\mathfrak{A}_{\delta,j}} u_{\delta,j} \rightharpoonup \Psi_{\delta,\mathfrak{A}_\delta} u_\delta$ weakly in $L^2(\Omega_\delta \times \Omega_\delta)$.*

- Prove $\lim_{j\to\infty} B_{\delta,\mathfrak{A}_{\delta,j}}(u_{\delta,j}, u_{\delta,j} - u_\delta) = 0$ as a consequence of G-convergence
- Prove $\lim_{j\to\infty} B_{\delta,\mathfrak{A}_{\delta,j}}(u_\delta, u_{\delta,j} - u_\delta) = 0$ by controlling $|\langle d_{k_\delta} \Psi_{\delta,\mathfrak{A}_{\delta,j}} u_{\delta,j}, u_{\delta,j} - u_\delta \rangle|$ with Hölder and using Nonlocal Integration by Parts
- Conclude that $u_{\delta,j} \to u_\delta$ strongly in $X(\Omega_\delta;\mathbb{R}^n)$
- Finally, prove desired weak convergence directly

Cost Functional Setup

Cost functional

$$J(\mathfrak{A}, u) \ := \ \int_{\Omega} F(x, u(x))dx + \iint_{\Omega\times\Omega} A(x, y, \mathfrak{A}(x, y))dxdy$$

where $F : \Omega_{\delta} \times \mathbb{R}^n \to \mathbb{R}$ satisfies the following rudimentary conditions:

1. $F(\cdot, v)$ is measurable for all $v \in \mathbb{R}^n$;
2. $F(x, \cdot)$ is lower semi-continuous for any fixed $x \in \Omega$.
3. There exists a constant $c_1 > 0$ and a function $\ell_1 \in L^1(\Omega)$ for which

   $$|F(x, v)| \leq c_1|v|^2 + \ell_1(x)$$

   holds for all $x \in \Omega$ and $v \in \mathbb{R}^n$.

Cost Functional Setup (continued)

We also say that $A : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies these conditions:

1. $A(\cdot, \cdot, w)$ is measurable for all $w \in \mathbb{R}$;
2. $A(x, y, \cdot)$ is weakly lower semi-continuous for any fixed $(x, y) \in \Omega \times \Omega$.
3. There exists another constant $c_2 > 0$ and a function $\ell_2 \in \ell^1(\Omega \times \Omega)$ for which

   $$|A(x, y, w)| \leq c_2 |w|^2 + \ell_2(x, y)$$

   holds for all $(x, y) \in \Omega \times \Omega$ and $w \in \mathbb{R}$.

NOTE: One option is $\iint_{\Omega\times\Omega} A(x, y, \mathfrak{A}(x, y))dxdy = \frac{\lambda}{2}\|\mathfrak{A}\|^2_{L^2(\Omega\times\Omega)}$

Non-local Optimal Design Problem Statement

Find $(\overline{\mathfrak{A}_\delta}, \overline{u_\delta}) \in \mathcal{H} \times X_0$ such that

$$J(\overline{\mathfrak{A}_\delta}, \overline{u_\delta}) \;=\; \min_{\mathfrak{A}_\delta \in \mathcal{H}, u_\delta \in X_0} \left\{ \int_\Omega F(x, u_\delta(x))dx + \iint_{\Omega\times\Omega} A(x, y, \mathfrak{A}_\delta(x, y))dxdy \right\},$$

The minimization is over pairs $(\mathfrak{A}_\delta, u_\delta) \in \mathcal{H} \times X_0(\Omega_\delta; \mathbb{R}^n)$ that satisfy

$$B_{\delta,\mathfrak{A}_\delta}(u_\delta, v) \;=\; \langle g, v\rangle, \text{ for all } \; v \in X_0(\Omega_\delta; \mathbb{R}^n),$$

Existence of nonlocal optimal designs

Theorem (Existence of Non-local Optimal Design)

Let $\delta > 0$ be fixed. There exists a pair $(\overline{\mathfrak{A}_\delta}, \overline{u_\delta})$ solving the non-local optimal design problem.

- Use direct method
- Use weak convergence of states to identify a limit [of minimizing sequence]
- Weak-* limit of minimizing sequence of controls belongs to $\mathcal{H}$
- Solutions need not be unique!

Compliance and Reduced Cost

We henceforth assume J is of a compliance form, i.e.

$$J(\mathfrak{A}_\delta, u_\delta) \; = \; \int_\Omega g(x) \cdot u_\delta(x) dx + \iint_{\Omega\times\Omega} A(x, y, \mathfrak{A}_\delta(x, y)) dxdy$$

Let $r_\delta : \mathcal{H} \to \mathbb{R}$ denote the reduced cost functional:

$$r_\delta(\mathfrak{A}_\delta) \; = \; \int_\Omega g(x) \cdot S^\delta(\mathfrak{A}_\delta) dx + \iint_{\Omega\times\Omega} A(x, y, \mathfrak{A}_\delta(x, y)) dxdy$$

Γ-convergence

Definition (Γ-convergence)

We say that the family of functionals $\{r_\delta\}_{\delta>0}$ **Γ-converges** to the functional r (written $r_\delta \xrightharpoonup{\Gamma_*} r$) with respect to the weak-* topology on $L^\infty(\Omega \times \Omega)$ if the following hold:

1. **The liminf property:** If $\{\mathfrak{A}_\delta\}_{\delta>0} \subset \mathcal{H}$ is a sequence such that $\mathfrak{A}_\delta \xrightharpoonup{*} \mathfrak{A}$ in the weak-* topology on $L^\infty(\Omega \times \Omega)$, then we have the lim-inf inequality
$$\operatorname{liminf}_{\delta\to 0^+} r_\delta(\mathfrak{A}_\delta) \;\geq\; r_0(\mathfrak{A}).$$
2. **Recovery sequence property:** If $\mathfrak{A} \in \mathcal{H}$, then there exists a recovery sequence $\{\mathfrak{A}_\delta\}_{\delta>0} \subset \mathcal{H}$ such that $\mathfrak{A}_\delta \xrightharpoonup{*} \mathfrak{A}$ in the weak-* topology on $L^\infty(\Omega \times \Omega)$ and
$$\operatorname{limsup}_{\delta\to 0^+} r_\delta(\mathfrak{A}_\delta) \;\leq\; r_0(\mathfrak{A}).$$

Γ-convergence result

Theorem (Γ-convergence of optimal designs)

We have that $r_\delta \xrightarrow{\Gamma} r_0$ *with respect to the weak-* topology on* $L^\infty(\Omega \times \Omega)$.

Corollary (Convergence of Minimizers for Optimal Design)

Let $\{\mathfrak{A}_\delta\}_{\delta>0} \subset \mathcal{H}$ *be such that* $\overline{\mathfrak{A}_\delta} \in argmin_{\mathfrak{A}_\delta \in \mathcal{H}} r_\delta(\mathfrak{A}_\delta)$, *and let* $\overline{\mathfrak{A}_0} \in \mathcal{H}$ *be such that* $\overline{\mathfrak{A}_0} \in argmin_{\mathfrak{A}_0 \in \mathcal{H}} r_0(\mathfrak{A}_0)$. *Then we have the convergence*

$$\lim_{\delta \to 0^+} r_\delta(\overline{\mathfrak{A}_\delta}) \; = \; r_0(\overline{\mathfrak{A}_0}).$$

Local design problem

Find $(\overline{\mathfrak{A}}, \overline{u}) \in \mathcal{H} \times H_0^1(\Omega; \mathbb{R}^n)$ such that

$$J(\overline{\mathfrak{A}}, \overline{u}) = \min_{\mathfrak{A} \in \mathcal{H}, u \in H_0^1} \left\{ \int_\Omega g(x) \cdot u(x) dx + \iint_{\Omega \times \Omega} A(x, y, \mathfrak{A}(x, y)) dxdy \right\}$$

over pairs $(\mathfrak{A}, u) \in \mathcal{H} \times H_0^1$ that satisfy

$$B_{0,\mathfrak{A}}(u, w) = \int_\Omega g(x) \cdot w(x) dx \qquad \forall w \in H_0^1$$

Theorem

There exists a solution $(\overline{\mathfrak{A}_0}, \overline{u_0})$ to the local optimal design problem.

Outline

1 Overarching Ideas

2 Notation

3 Optimal Control: Analysis

4 Optimal Control: Discretization

5 Optimal Design: Analysis

6 Closing Remarks

Summary

- Well-posedness of control problem, existence of minimizers for design problem
- Convergence of minimizers as $\delta \to 0^+$
- Finite element approximation of control problem
- Asymptotic compatibility for control problem

What's next?

- Finish simulations for control problem
- Finite element approximation for design problem
- Asymptotic compatibility results for design problem

ArXiV version of control paper: `https://arxiv.org/pdf/2304.09328.pdf`