MATH 574 PRESENTATION NOTES: Γ-CONVERGENCE IN FINITE ELEMENTS AND CALCULUS OF VARIATIONS

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CONTENTS

1. WHAT IS THIS?

For MATH 574: Finite Element Methods in Fall 2021 at the University of Tennessee, I was asked to give a presentation on a topic that extended the course material to earn my course grade. This document comprises the notes I prepared for that presentation.

2. INTRODUCTION

2.1. **Quadratic Energies.** This example will help motivate the significance of Γ-convergence. Consider the quadratic functional

$$J(u) := \frac{1}{2}\mathcal{A}(u,u) - F(u) \tag{2.1}$$

on a Hilbert Space H, with $\mathcal{A}$ a bounded, coercive bilinear form[1] on $H \times H$ and F a linear form on H. As is typical with a Ritz method[2], let $\{H_N\}_{N=1}^{\infty}$ be a collection of subspaces of H so that

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[1]With boundedness constant M and coercivity constant α

[2]of which Galerkin methods are a special case

$\dim(H_N) = N$ for all $N \in \mathbb{N}^+$. With the assumptions on J, there will be a unique minimizer u of J on H, and also a unique minimizer u_N of J on H_N for each $N \in \mathbb{N}^+$.

Also assume that these subspaces are dense in the limit. That is, assume

$$\overline{\bigcup_{N \in \mathbb{N}^+} H_N} \;=\; H. \tag{2.2}$$

Then we will show two things:

Claim A: $J(u_N) \to J(u)$ as $N \to \infty$

Claim B: $u_N \to u$ as $N \to \infty$

Due to (2.2), we can select (up to a sub-sequence of the sets in the union (2.2) that is not relabeled) $v_N \in H_N$ such that $v_N \to u$ as $N \to \infty$. Then for each N we have that

$$J(u) \;\leq\; J(u_N) \;\leq\; J(v_N), \tag{2.3}$$

so to complete the proof of Claim A it suffices to show that $J(v_N) \to J(u)$ by the Squeeze Theorem. Clearly, $F(v_N) \to F(u)$. Moreover, boundedness and coercivity of $\mathcal{A}$ give constants M and α such that

$$\alpha\|v_N - u\|_H^2 \leq\; \mathcal{A}(v_N - u, v_N - u) \;\leq\; M\|v_N - u\|_H^2 \tag{2.4}$$

Upon expanding $\mathcal{A}(v_N - u, v_N - u)$, and recalling the convergence properties we have, we may send $N \to \infty$ above, treating the bilinear form as a norm equivalent to $\|\cdot\|_H$, and we obtain that $\mathcal{A}(v_N, v_N) \to \mathcal{A}(u, u)$, completing the proof of Claim A.

Now we prove Claim B, using Claim A. Furthermore, for any $v \in H$, our functional being quadratic gives us

$$J(v) - J(u) - J'(u)(v - u) \;=\; \frac{1}{2}\mathcal{A}(v - u, v - u) \;\geq\; \frac{\alpha}{2}\|v - u\|_H^2, \tag{2.5}$$

and since $J'(u) = 0$, upon replacing $v := u_N$ to get

$$J(u_N) - J(u) \;\geq\; \frac{\alpha}{2}\|u_N - u\|_H^2. \tag{2.6}$$

Finally, sending $N \to \infty$ and using Claim A gives us Claim B.

Remark 2.1. *We only used the specific structure of our functional to prove Claim B.*

2.2. **The Lavrentiev Gap Phenomenon.** The purpose of this example is to show that the convergence of minimizers is not guaranteed by simply assuming that the approximation spaces are dense in the full space. Other conditions are needed, and these will be revealed in due time. In any case here are the details of the example.

Example 2.2 ([Bar] Example 4.1)**.** *Let our admissible class be* $X := W_0^{1,1}(0,1)$*. Consider the functional*

$$I(u) \;:=\; \int_0^1 (x - u(x)^3)^2 |u'(x)|^6 dx. \tag{2.7}$$

For each $h > 0$, let $\mathscr{T}_h$ be a triangulation (partition) of $(0,1)$ and define $X_h := X \cap \mathscr{S}^1(\mathscr{T}_h)$. Then the function $u(x) := x^{\frac{1}{3}}$ is a minimizer for I in X, but for each $h > 0$, we have the gap inequality

$$0 = \min_{u \in W_0^{1,1}(0,1)} I(u) < \min_{u \in W_0^{1,1}(0,1) \cap W^{1,\infty}(0,1)} I(u) \leq \min_{u_h \in X_h} I(u_h). \tag{2.8}$$

That is, even if we send $h \to 0^+$ above, only the last part of the inequality depends on h, so these minima expressions cannot converge to 0! The strictness of the first inequality is crucial.

2.3. **Key Definitions.**

Definition 2.3 (Types of Convergence and Limiting Behavior)**.** *Let $(X, \|\cdot\|_X)$ be a Banach space, $I : X \to \mathbb{R}$ be a functional.*

*a. A sequence $\{x_k\}_{k=1}^{\infty} \subset X$ is said to **weakly converge** to an $x \in X$, if for each $y \in X^*$, necessarily $\lim_{k\to\infty} y(x_k) = y(x)$. Denote this as $x_k \rightharpoonup x$.*

*b. A sequence $\{x_k\}_{k=1}^{\infty} \subset X$ is said to **strongly converge** to $x \in X$ if $\|x_k - x\|_X \to 0$ as $k \to \infty$*

*c. The functional I is said to be **weakly lower semi-continuous** if whenever $x_k \rightharpoonup x$, we have $I(x) \leq \text{liminf}_{k\to\infty} I(x_k)$.*

*d. The functional I is said to be **[strongly] lower semi-continuous** if whenever $x_k \to x$, we have $I(x) \leq \text{liminf}_{k\to\infty} I(x_k)$.*

*e. The functional I is said to be **[strongly] continuous** if whenever $x_k \to x$, we have $I(x) = \lim_{k\to\infty} I(x_k)$.*

*f. The functional I is said to be **coercive** if whenever x_k is a sequence with $\|x_k\| \to \infty$, necessarily $I(x_k) \to \infty$ as $k \to \infty$.*

Definition 2.4 (Quasiconvexity, [Rin] Equation (5.2))**.** *A locally bounded, Borel-measurable function $h : \mathbb{R}^{m\times d} \to \mathbb{R}$ is **quasiconvex** if*

$$h(A) \leq \fint_{B(0,1)} h(A + \nabla \psi(z)) dz \tag{2.9}$$

for all $A \in \mathbb{R}^{m\times d}$ and all $\psi \in W_0^{1,\infty}(B(0,1); \mathbb{R}^m)$.

Remark 2.5. *The previous definition holds even for $m = 1$ and $d = 1$. Moreover, all convex functions are automatically quasiconvex, and the notions of convex and quasiconvex are equivalent for scalar-valued functions.*

Definition 2.6 (Γ-convergence)**.** *Let X be a Banach space, $I : X \to \mathbb{R} \cup \{+\infty\}$ and let $\{I_h\}_{h>0}$ be a sequence of functionals $I_h : X \to \mathbb{R} \cup \{+\infty\}$. This sequence **Γ-converges** to I as $h \to 0$, denoted by $I_h \xrightarrow{\Gamma} I$ with respect to a topology (or norm) on X if the following conditions hold:*

*i) **Liminf inequality:** for every sequence $\{u_h\}_{h>0} \subset X$ with $u_h \to u$ for a $u \in X$, then*

$$I(u) \leq \text{liminf}_{h\to 0^+} I_h(u_h). \tag{2.10}$$

*ii) **Recovery sequence:** for every $u \in X$ there exists a sequence $\{u_h\}_{h>0} \subset X$ with $u_h \to u$ and*

$$\lim_{h\to 0^+} I_h(u_h) \;=\; I(u). \tag{2.11}$$

Remark 2.7. *Condition ii) can be replaced by the following condition, which seems weaker but coupled with i) is actually equivalent: for every $u \in X$ there exists a sequence $\{u_h\}_{h>0} \subset X$ with $u_h \to u$ and*

$$limsup_{h\to 0^+} I_h(u_h) \;\le\; I(u). \tag{2.12}$$

3. Facts about Γ-convergence

3.1. **Assorted Facts.** Here we present some basic facts about Γ-convergence.

Lemma 3.1 (Exercise in [Rin])**.** *If a sequence of functionals $\{I_h\}_{h>0}$ has a Γ-limit I, then that limit is uniquely determined.*

Proposition 3.2 (Proposition 13.2 [Rin])**.** *If $I_h : X \to \mathbb{R} \cup \{+\infty\}$ is a sequence of functionals that Γ-converges to $I : X \to \mathbb{R} \cup \{+\infty\}$, then I is lower semi-continuous.*

Corollary 3.3. *Let X be a normed space*[3]*. If $I_h : X \to \mathbb{R} \cup \{+\infty\}$ is a sequence of functionals that Γ-converges to $I : X \to \mathbb{R} \cup \{+\infty\}$ and I is a convex functional, then I is weakly lower semi-continuous.*

Proof. This follows from Proposition 3.2 and the fact that any functional that is convex and lower semi-continuous is automatically weakly lower semi-continuous. □

3.2. **Conforming Discretizations.**

Definition 3.4 (Confirming Subspaces)**.** *We say that a sequence of subspaces $\{X_h\}_{h>0}$ of X used in a finite element approximation* ***conform*** *to X if*

$$I_h(u_h) \;=\; I(u_h) \tag{3.1}$$

for all $u_h \in X_h$ and all $h > 0$.

This type of discretization is introduced because it aids in a sufficient condition for Γ-convergence of a discretization.

Theorem 3.5 (Theorem 4.1 in [Bar])**.** *Let $\{X_h\}_{h>0}$ be a family of conforming subspaces for the Banach space X that are also dense [in the limit] in X with respect to the strong [normed] topology of X. If I is weakly lower semi-continuous and strongly continuous, then $I_h \xrightarrow{\Gamma} I$ as $h \to 0$ with respect to weak convergence in X.*

Proof. Choose a sequence $\{u_h\}_{h>0} \subset X$ and a $u \in X$ for which $u_h \rightharpoonup u$ as $h \to 0^+$. Since we have a conforming discretization,

$$I_h(u_h) \;\ge\; I(u_h) \tag{3.2}$$

[3]to assure that we have a notion of strong convergence that is meaningful

for all $h > 0$, so the weak lower semi-continuity of I gives us the lim-inf inequality, namely

$$\text{liminf}_{h\to 0^+} I_h(u_h) \;\geq\; \text{liminf}_{h\to 0^+} I(u_h) \;\geq\; I(u). \tag{3.3}$$

Now we prove existence of a recovery sequence. For a given $u \in X$, pick $\{u_h\}_{h>0}$ such that $u_h \in X_h$ for all $h > 0$ and $u_h \to u$ in X. Then, since I is strongly continuous and $I_h(u_h) = I(u_h)$ for all $h > 0$, taking a limit yields

$$I(u) \;=\; \lim_{h\to 0^+} I_h(u_h). \tag{3.4}$$

□

4. WHY DISCRETIZE?

The notion of Γ-convergence is one that can be proven without any introduction of finite elements, if one considers the exact form of Definition 2.6. However, it is often of interest in finite element methods, and numerical analysis in general, to obtain convergence rates. That is, do there exist $C, \beta > 0$ such that

$$\left| \min_{u_h\in X_h} I_h(u_h) - \min_{u\in X} I(u) \right| \;\leq\; Ch^\beta ? \tag{4.1}$$

This example illustrates one problem where we are able to obtain a rate of convergence.

Example 4.1. *Let $X := H_0^1(\Omega)$ and $X_h := \mathscr{S}_0^1(\mathscr{T}_h)$ for a family of triangulations $\{\mathscr{T}_h\}_{h>0}$ of Ω. Let $f \in L^2(\Omega)$ and $g \in L^2(\Gamma_N)$, and define the functional*

$$I(u) \;:=\; \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega f u dx - \int_{\Gamma_N} g u ds. \tag{4.2}$$

Here Γ_N is the ***Neumann boundary****, which is the portion of the boundary on which a Neumann boundary condition is prescribed.*

Also define $I_h : H_0^1(\Omega) \to \mathbb{R}$ so that $I_h \equiv I$ on $\mathscr{S}_0^1(\mathscr{T}_h)$. The Dirichlet Energy refers to the first term in the functional. Those are weakly lower semicontinuous and strongly continuous. Meanwhile, the other terms are weakly continuous on $H_0^1(\Omega)$. Finally, the subspaces we chose are dense in the limit towards $H_0^1(\Omega)$, so by Theorem 3.5, we have that $I_h \xrightarrow{\Gamma} I$ as $h \to 0^+$.

Now we can consider convergence rates. Recall the abstract Ceá's Lemma we proved in class: when we have a coercive, bounded bi-linear form and a bounded linear form on a Hilbert space H and u_N represents the minimizer of our functional on a subspace $H_N \subset H$ of dimension N, we have the ***quasi-best approximation property***

$$\|u - u_N\|_H \leq \frac{M}{\alpha} \inf_{v_N\in H_N} \|u - v_N\|_H, \tag{4.3}$$

where M is the boundedness constant of the bilinear form, and α is the coercivity constant. For our problem, this lemma takes the form

$$\|\nabla(u - u_h)\|_{L^2(\Omega)} \leq \inf_{v_h\in \mathscr{S}_D^1(\mathscr{T}_h)} \|\nabla(u - v_h)\|_{L^2(\Omega)}. \tag{4.4}$$

Then due to error interpolation on the affine function space, we obtain

$$\|\bigtriangledown(u-u_h)\|_{L^2(\Omega)} \leq ch^2[u]_{H^2(\Omega)}. \tag{4.5}$$

Now we estimate $|I(u)-I_h(u_h)|$ with the Triangle Inequality to obtain the terms that we want:

$$|I(u)-I_h(u_h)| \leq \frac{1}{2}\left|\int_\Omega |\bigtriangledown u|^2 - |\bigtriangledown u_h|^2 dx\right| + \left|\int_\Omega f(u-u_h)dx\right| + \left|\int_{\Gamma_N} g(u-u_h)dS\right|. \tag{4.6}$$

Owing to Cauchy-Schwarz, we then get

$$|I(u)-I_h(u_h)| \leq \frac{1}{2}\left|\int_\Omega |\bigtriangledown u|^2 - |\bigtriangledown u_h|^2 dx\right| + \|f\|_{L^2(\Omega)}\|u-u_h\|_{L^2(\Omega)} + \|g\|_{L^2(\Gamma_N)}\|u-u_h\|_{L^2(\Gamma_N)}. \tag{4.7}$$

Now, we can handle the $\|u-u_h\|_{L^2(\Gamma_N)}$ term by means of the trace inequality:

$$|I(u)-I_h(u_h)| \leq \frac{1}{2}\left|\int_\Omega |\bigtriangledown u|^2 - |\bigtriangledown u_h|^2 dx\right| + \|f\|_{L^2(\Omega)}\|u-u_h\|_{L^2(\Omega)} + C_{tr}\|g\|_{L^2(\Gamma_N)}\|u-u_h\|_{W^{1,2}(\Omega)}. \tag{4.8}$$

Since we want factors of $\|\bigtriangledown(u-u_h)\|_{L^2(\Omega)}$, we look to use the Poincaré Inequality on the lower-order terms to condense them all as

$$|I(u)-I_h(u_h)| \leq \frac{1}{2}\left|\int_\Omega |\bigtriangledown u|^2 - |\bigtriangledown u_h|^2 dx\right| + C\|\bigtriangledown(u-u_h)\|_{L^2(\Omega)}. \tag{4.9}$$

Now, by means of treating the remaining integral as a difference of squares, we obtain the inequality

$$|I(u)-I_h(u_h)| \leq \frac{1}{2}\|\bigtriangledown u + \bigtriangledown u_h\|_{L^2(\Omega)}\|\bigtriangledown u - \bigtriangledown u_h\|_{L^2(\Omega)} + C\|\bigtriangledown(u-u_h)\|_{L^2(\Omega)}. \tag{4.10}$$

The factor $\|\bigtriangledown u + \bigtriangledown u_h\|_{L^2(\Omega)}$ is uniformly bounded as $h \to 0^+$, so in essence this becomes

$$|I(u)-I_h(u_h)| \leq C\|\bigtriangledown(u-u_h)\|_{L^2(\Omega)}. \tag{4.11}$$

Finally, we can use the standard error estimate $\|\bigtriangledown u - \bigtriangledown u_h\|_{L^2(\Omega)} \leq ch[u]_{H^2(\Omega)}$ to get the desired estimate of[4]

$$|I(u)-I_h(u_h)| \leq Ch. \tag{4.12}$$

[4]The constants change at each step!

5. Application: Hyperelasticity

Example 5.1 (Hyperelasticity Problem). *Let our admissible class be* $X := \{y \in W^{1,2}(\Omega;\mathbb{R}^d), y|_{\Gamma_D} = \widetilde{y}|_{\Gamma_D}\}$ *where* $\widetilde{y}_D \in W^{1,2}(\Omega;\mathbb{R}^d)$ *is Dirichlet boundary data. Assume that* $W : \mathbb{R}^{d\times d} \to \mathbb{R}$ *is a continuous and quasi-convex mapping with growth bounds*

$$-c_1 + c_2|F|^2 \leq \ W(F) \ \leq \ c_1 + c_2|F|^2. \tag{5.1}$$

Let $f \in L^2(\Omega;\mathbb{R}^d)$ *and* $g \in L^2(\Gamma_N;\mathbb{R}^d)$ *be our data, and define our functional in terms of it:*

$$I(u) \ := \ \int_\Omega W(\bigtriangledown u)dx - \int_\Omega f \cdot u dx - \int_{\Gamma_N} g \cdot u dS \tag{5.2}$$

We will show that this functional is weakly lower semi-continuous and coercive on $W^{1,2}(\Omega;\mathbb{R}^d)$. *Theorem 2.2 of* [Bar] *says that quasi-convex functionals are automatically weakly lower semicontinuous in* $W^{1,p}(\Omega;\mathbb{R}^d)$*, and we will accept this without proof.*

Suppose now that $\{u_j\}_{j=1}^\infty \subset W^{1,2}(\Omega;\mathbb{R}^d)$ *is a sequence converging strongly to some* $u \in W^{1,2}(\Omega;\mathbb{R}^d)$. *Then there is a sub-sequence* $\{u_{j_k}\}_{k=1}^\infty$ *of* $\{u_j\}_{j=1}^\infty$ *such that* $\bigtriangledown u_{j_k} \to \bigtriangledown u$ *for a.e.* $x \in \Omega$. *Then for this sub-sequence we may use the Generalized Dominated Convergence Theorem to conclude that*

$$\lim_{k\to\infty} \int_\Omega W(\bigtriangledown u_{j_k}(x))dx \ = \ \int_\Omega W(\bigtriangledown u(x))dx. \tag{5.3}$$

With these convergence properties in mind, one can conclude that we have Γ*-convergence with an appropriate choice of limiting sub-spaces. If* y_D *is piecewise affine boundary data, then the set* $X_h := X \cap \mathscr{S}^1(\mathscr{T}_h)^d$ *is nonempty, and the finite element spaces are dense* [5].

References

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[5] see Lemma 3.6 in [Bar] for a proof for the $m = 1$ case, which easily generalizes to higher dimensions