MATH 113 PRACTICE PROBLEMS: SPRING 2022

1. Notes

These problems are provided for your practice. They are not arranged in any particular order of difficulty. They do not necessarily represent all course material. Also, remember to consider some additional places for problems: old homework assignments and old stretch activities. It's important to try as many as possible.

Any problem labeled "student problem" was created by a previous or current student of mine, and I wish to acknowledge that without revealing the students' identities.

Finally, this document will be updated as I either think of more problems to add, or as typos are found.

2. Sequences and Recursions

2.1. **Problem 1.** Explain in words the difference between a recursive formula and a non-recursive formula. Consider the sequence of perfect squares

$$1, 4, 9, 16, 25, \ldots$$

Identify both a recursive formula and a non-recursive formula for this sequence.

2.2. **Problem 2.** Explain in words the pattern in the sequence

$$4, 13, 40, 121, 364, \ldots$$

What is the relationship between the "previous" term and the "next" term? Use this explanation to find a recursive formula.

2.3. **Problem 3.** Does the recursive formula

$$S_{n+1} = 6S_n - 3$$

satisfy the non-recursive formula

$$S_n = 6n + 1?$$

2.4. **Problem 4.** Does the recursive formula

$$S_{n+1} = 6S_n + 1$$

satisfy the non-recursive formula

$$S_n = 6n + 1?$$

2.5. **Problem 5.** Explain in words the pattern in the sequence

$$5, 11, 23, 47, \ldots$$

What is the relationship between the "previous" term and the "next" term? What will the next two numbers in the sequence be? Determine a recursive formula for the sequence.

2.6. **Problem 6.** Go back to the table in the NUMBER: Day 1 Story on Slide 16. Fill out what the next three rows in the table would be. That is, calculate the entries in the table using the recursive formula *and* the non-recursive formula.

2.7. **Problem 7.** Go back to the quilts in the NUMBER: Day 1 Story on Slide 29. Find a recursive and a non-recursive formula for the number of blue squares in the pattern.

2.8. **Problem 8.** If $S_n = n^2 + 1$ and $T_n = n^3 + 1$, find $S_2 + T_2$.

2.9. **Problem 9.** Consider the sequence where $A_1 = 0$ and $A_n = 2n + 1$ for $n \geq 2$.
i) Is this recursive or not recursive?
ii) Find A_2, A_3, A_4, A_5, A_6.
iii) Which terms are even, and which ones are odd? Explain your reasoning.

2.10. **Problem 10.** Consider the sequence where $A_1 = 0$ and $A_n = 2A_{n-1} + 1$ for $n \geq 2$.
i) Is this recursive or not recursive?
ii) Find A_2, A_3, A_4, A_5, A_6.
iii) Which terms are even, and which ones are odd? Explain your reasoning.

2.11. **Problem 11.** Calculate a handful of terms for the sequence where $A_1 = 0$ and $A_n = 2A_{n-1} + 1$ for $n \geq 2$. What pattern arises?

2.12. **Problem 12.** Explain in words the pattern in the sequence

$$7, 12, 22, 42, 82$$

and provide a recursive formula for this sequence.

2.13. **Problem 13.** While discussing Problem 5 in class, a student suggested an alternative recursive formula for the sequence

$$5, 11, 23, 47, ...$$

That formula is

$$S_n = 3S_{n-1} - 2S_{n-2}; S_1 = 5, S_2 = 11$$

Verify that this recursive formula gives the same next two terms for the sequence as the following recursion does:

$$S_n = 2S_{n-1} + 1$$

In other words, I am suggesting that there are two distinct recursive formulas describing the same sequence!

3. Sets of Numbers

3.1. **Problem 1.** Is the statement “every whole number is an integer" true or false? If it is false, give a counterexample.

3.2. **Problem 2.** Is the statement “every integer is a whole number" true or false? If it is false, give a counterexample.

3.3. **Problem 3.** Is the statement "the sum of any two rational numbers is rational" true or false? If it is false, give a counterexample.

3.4. **Problem 4.** List five different irrational numbers. Get into the habit of doing this whenever you are asked if a statement about rational and irrational numbers is true or false.

3.5. **Problem 5.** On Day 4 we created a one-to-one correspondence between the natural numbers and the whole numbers, using a diagram with arrows pointing between the two sets. Brainstorm and try to create some *different* one-to-one correspondences between the natural numbers and the whole numbers.

3.6. **Problem 6.** Create a one-to-one correspondence between the natural numbers $\{1, 2, 3, 4, \dots\}$ and the even natural numbers $\{2, 4, 6, 8, \dots\}$.

3.7. **Problem 7.** In the story for Day 3 we saw that the product of two irrational numbers is not always irrational, and provided a few counterexamples. Try to come up with two or three new ones.

3.8. **Problem 8.** Consider the set of all two-letter words (any two letters of the English alphabet listed back-to-back). Is this set finite, countably infinite (Aleph Naught), or uncountably infinite (larger than Aleph Naught)? Explain your answer.

3.9. **Problem 9 (student problem).** This is a slightly more difficult problem that sprang up as part of a review day discussion. Is the following statement true or false?

A prime number can never be perfect

Explain your reasoning if true, or provide a counterexample if false (i.e. find a prime number that IS perfect)

3.10. **Problem 10.** Sort these lists of numbers from smallest to biggest:

Whole Numbers, Natural Numbers, Integers, Rational Numbers

3.11. **Problem 11.** Are all natural numbers whole numbers? If so, explain why; if not, give a counterexample.

3.12. **Problem 12.** Are all rational numbers integers? If so, explain why; if not, give a counterexample.

4. Divisibility and Primes

4.1. **Problem 1.** a. Every prime number is a natural number. Does that mean the size of the set of prime numbers is smaller than or larger than the set of natural numbers?

b. In class we showed that there are infinitely many prime numbers. Is the set of prime numbers $\aleph_0$? Explain why or why not.

4.2. **Problem 2.** a. Find a composite natural number that has exactly one prime factor.

b. Find a composite natural number that has exactly two prime factors.

Remember, 1 is NOT a prime number.

4.3. **Problem 3.** Remember that Goldbach's Conjecture claims that every even number greater than 2 can be written as the sum of two prime numbers. Here we'll do some further discussion of this conjecture.

a. Write 4 as the sum of two prime numbers. Is that the only way you can do it? What is so special about your answer?

b. Write 22 as the sum of two prime numbers. How many ways can you do it?

c. Write 94 as the sum of two prime numbers. *Hint:* Go back to the Sieve of Eratosthenes in the NUMBER: Day 5 Story on Slide 7. One of the boxed numbers will be very helpful.

4.4. **Problem 4.** a. Is 37 divisible by 3? What is the remainder?

b. Is 36 divisible by 3? What is the remainder?

c. Let $P = (2 \cdot 3 \cdot 5 \cdot 7 \cdot 11) + 1$. If you divide P by any of $2, 3, 5, 7, 11$, what is the remainder?

4.5. **Problem 5. Cousin primes** are prime numbers that differ by 4. Subtract them, and you will get 4. List two examples of pairs of cousin primes.

4.6. **Problem 6.** Describe, in your own words, the difference between a least common multiple (LCM) and greatest common divisor (GCD).

4.7. **Problem 7.** Find the LCM and GCD of 24 and 36. Show the Venn Diagram you make.

4.8. **Problem 8.** Find the LCM and GCD of 60 and 32. Show the Venn Diagram you make.

4.9. **Problem 9.** Pick any two prime numbers that are different from each other. Make a Venn Diagram as if you were to find the LCM and the GCD. Are there any numbers in the middle? Why not? After considering this, find the LCM and GCD of the two prime numbers you chose.

4.10. **Problem 10.** Which of the following is the GCF for 12 and 72?

A. 1
B. 12
C. 24
D. 72

4.11. **Problem 11.** i) State the Twin Primes Conjecture.
ii) Give 3 examples of pairs of twin primes.
iii) Give an example of two prime numbers that are NOT twin primes.
iv) State a NEW conjecture about prime numbers. Be creative!
v) Are the prime numbers of size aleph naught? ($\aleph_0$) (Hint: remember first the definition of what it means for a set to be of size aleph naught)

5. Fibonacci Numbers and the Golden Ratio

5.1. **Problem 1.** Calculate the following:
a.
$$F_4 + F_3 + F_2 + F_1$$
b.
$$F_7 - F_5$$
c.
$$F_{12} - F_{10}$$

5.2. **Problem 2.** Let H_n be a sequence where $H_1 = 1$, $H_2 = 4$, and for $n > 2$ satisfies the recurrence relation
$$H_n = H_{n-1} + H_{n-2}$$
Find H_3, H_4, H_5, H_6. Also find $H_9 - H_7$.

5.3. **Problem 3.** List out some Fibonacci Numbers. Circle the ones that are divisible by 4. Do you see a pattern? Do the same thing for the Fibonacci Numbers divisible by 3.

5.4. **Problem 4.** Transform
$$\psi = \cfrac{2}{2 + \cfrac{2}{2 + \cfrac{2}{\cdots}}}$$
into a quadratic equation involving ψ, using telescoping. You don't have to solve it. NOTE: quadratic means that ψ^2 appears somewhere. (Hint: look at Day 8 Story, Slides 10-14)

5.5. **Problem 5.** Explain why the Golden Ratio $\phi = \dfrac{1+\sqrt{5}}{2}$ is an irrational number.

5.6. **Problem 6.** Go back to Slide 5 on the Day 8 Story. True or false: all of the numbers plotted on the graph are irrational.

5.7. **Problem 7.** Rewrite this equation using telescoping. Do not solve the equation.
$$x = 3 + \sqrt{3 + \sqrt{3 + \sqrt{3 + \ldots}}}.$$

6. Mathematical Modeling

6.1. **Problem 1.** Go back to Day 9 Story, Slide 10. Explain why the following proportion means the same thing as the one on the slide:

$$\frac{8800 \text{ pounds}}{40 \text{ cm}} = \frac{x \text{ cm}}{175 \text{ pounds}}$$

6.2. **Problem 2.** Using a simpler mathematical model as opposed to a more complicated model may be easier to use. List some *disadvantages* that might arise from using a simpler model.

6.3. **Problem 3.** Explain using words the difference between a linear model and an exponential model. There is more than one correct thing you can say here. We talked about this in the Day 9 Story, Slide 8.

6.4. **Problem 4.** Simplify the following expressions and write your final answers in scientific notation.

a. $\dfrac{5.6 \times 10^8}{2.8 \times 10^6}$

b. $\dfrac{5.6 \times 10^8}{2.8 \times 10^{-6}}$

c. $(3.7 \times 10^{-12})(2.1 \times 10^4)$

d. $(1.5 \times 10^6)(9.1 \times 10^3)$

6.5. **Problem 5.** Go back to Day 10 Story, Slide 3. We had said in class that the Carina Nebula is $7.57 \times 10^1 9$ meters from earth, and the distance from one end of this nebula to the other is 4.35×10^{18} meters. If you could copy the Carina Nebula and lay the copies down side by side in a line, how many copies could you fit before reaching Earth? (including the original)

6.6. **Problem 6 (student problem).** True or false? You can use proportional reasoning to compare objects using multiplicative thinking.

6.7. **Problem 7 (student problem).** A tiger is 96 inches tall standing up and has a tail length of 43.2 inches long. If a bear is 108 inches tall standing up, how long would its tail be if it was like a tiger's?

6.8. **Problem 8 (student problem).** Solve this proportion for x: $\dfrac{3}{30} = \dfrac{x}{60}$

A. 3
B. 8
C. 6
D. 28

6.9. **Problem 9 (student problem).** An NBA basketball player has is 7 feet tall and has feet that are 40 cm wide. If you are 5 feet tall and have feet like an NBA player's, how wide are your feet?

6.10. **Problem 10 (student problem).** If your neck was as long as an ostrich, how long would it be? The average neck length of an ostrich is 3.28 feet long* Assume you are 6 feet tall and an ostrich is 8 feet tall for this problem. Write your final answer in feet.

6.11. **Problem 11 (student problem).** True or false: proportional reasoning involves thinking about relationships and making comparisons of quantities or values.

6.12. **Problem 12 (student problem).** If a 5-foot person had the same height to weight ratio as a giraffe that was 18 feet tall and weighed 1700 pounds, how much would the person weigh? Round to the nearest pound.

6.13. **Problem 13 (student problem).** Which of the following are everyday use(s) of proportional reasoning?
A. Cooking
B. Art
C. Architecture
D. Currency exchange
E. All of the above

6.14. **Problem 14 (student problem).** Architects are making a scale model of a building. The height of the building is 10 ft and the height of the scale model is 1 ft. The width of the building is 25 ft. Find the width of the scale model.
A. 1.5 feet
B. 2.0 feet
C. 2.5 feet

6.15. **Problem 15 (student problem).** If you are 5 and a half feet tall and had the same height to weight ratio as a killer whale how much would you weigh? Keep in mind that a killer whale is 23 feet long and $12,000$ pounds. Round your answer to the nearest hundredth of a pound.

7. Angles and Side Lengths

7.1. **Problem 1.** Suppose a triangle has two angles of measure 60 and 70 degrees. What must be the measure of the third angle?

7.2. **Problem 2.** Suppose a quadrilateral has angles of measures 120, 80, and 75. What is the measure of the fourth angle?

7.3. **Problem 3.** A right triangle has one angle of measure 90 degrees. What is the maximum possible value of the product of the other two angles? (Hint: use trial and error; pick different values for the angles and multiply them together)

7.4. **Problem 4.** Go back to Slide 19 of SHAPE Day 1 Story. Try to find another order to find all of the unknown angle measures.

7.5. **Problem 5.** Can a triangle have two right angles? Why or why not?

7.6. **Problem 6.** Work through the calculations on Slides 20-23 of SHAPE Day 2 Story. Make sure you understand where all the equations come from.

7.7. **Problem 7.** Two side lengths of a right triangle are 3 and 5. What are the possible side lengths of the third side?

7.8. **Problem 8.** A right triangle has legs of length 6 and 8. What is the length of the hypotenuse?

7.9. **Problem 9.** A triangle has side lengths 4, 7, and 9. Is it a right triangle? How can you tell?

7.10. **Problem 10.** a. Triangles ABC and DEF are similar. Side AB is 6 meters long, BC is 9 meters long, and DE is 12 meters long. How long is EF?
b. Can you determine the length of AC or DF, or do you need more information?

7.11. **Problem 11.**

8. Circles

8.1. **Problem 1.** What is the area of a circle with radius 4 cm?

8.2. **Problem 2.** What is the area of a circle with diameter 10 cm?

8.3. **Problem 3.** What is the radius of a circle with area 36π square cm?

8.4. **Problem 4.** Take a circle of radius 12 cm, and cut out a square from the middle of the circle with side length 2 cm. What is the area of the figure?

8.5. **Problem 5 (student problem).** How many times larger is the area of a circle with a radius of 10 ft than a circle with a radius of 8 ft? round to two decimal places.

9. Surface Area and Volume, Prisms, Spheres, Pyramids

9.1. **Problem 1.** Find the surface area and volume of a cube with side length 5.

9.2. **Problem 2.** Find the surface area and volume of a rectangular prism with side lengths $2, 5$, and 9.

9.3. **Problem 3.** Can a cube have equal surface area and volume? If so, what would the side length have to be? *Hint:* take formulas for surface area and volume and set them equal to each other. You'll get an equation you can solve for the side length. Ignore units for this problem.

9.4. **Problem 4.** Your new swimming pool is a cylinder with radius 20 feet and depth 5 feet.
a. If you need tile around the sides and the bottom, how many square feet of tile do you need?
b. What is the volume of water needed to fill the pool?

9.5. **Problem 5.** Explain Cavalieri's Principle in your own words.

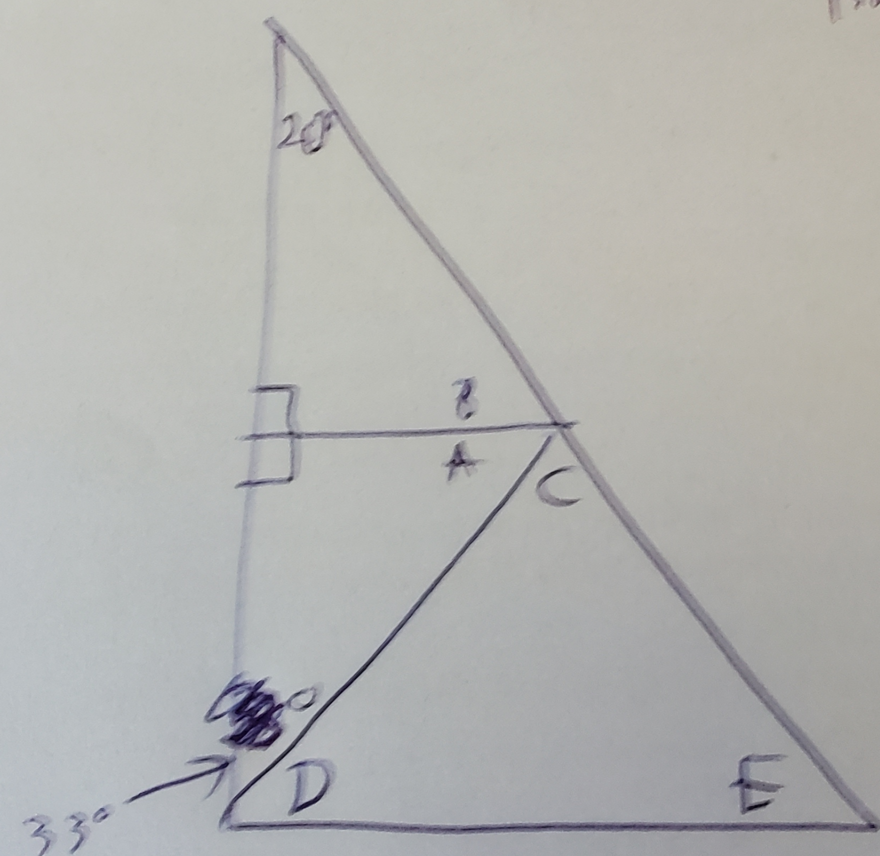
9.6. **Problem 6.** What other objects could you place in water to measure [the volume of] with volume displacement?

Problem 2 (copy)

The

is a

Find



9.7. **Problem 7.** You get ready to buy some fish. You pull out an aquarium tank that is a rectangular prism, 2 feet wide and 3 feet long. You fill it up with water until the level of the water is 9 inches from the bottom of the tank. Then you go and buy some fish from the pet store, and put them into the tank. The water level rises to 11 inches from the bottom of the tank. What is the total volume of all the fish? **NOTE:** be careful with the units!

9.8. **Problem 8.** Go back to Slide 17 from the SHAPE Day 5 Story. Explain how you would find the volume of one of the slanted pyramids. You don't have to actually carry out the calculation (you don't have enough information anyway).

9.9. **Problem 9.** Design a problem that is a variation of Problem 7, where the fish tank is a prism that is NOT a rectangular prism (pick a different shape!). Use whatever numbers you find suitable, then solve the new problem.

9.10. **Problem 10.** The diameter of the Earth is $7,917.5$ miles. Suppose global warming causes water levels to rise throughout much of the world, and that $10,000$ years from now, the total percentage of the Earth's surface that is covered with water is 77 percent. What surface area (in square miles) will not be covered in water $10,000$ years from now? (Hint: go back to SHAPE Day 6 Story Slide 19 for a similar problem)

9.11. **Problem 11.** Find the surface area of a sphere of radius 9.

9.12. **Problem 12.** Find the surface area of a sphere with diameter 12.

9.13. **Problem 13.** Consider a sphere with an unknown radius. Let S be a variable for the sphere's surface area. Let A be a variable for the area of a circle with the same center as the sphere and the circumference is on the surface of the sphere. Find $\dfrac{S}{A}$ (Hint: draw a picture)

9.14. **Problem 14.** What is the volume of a half-sphere with radius 6?

9.15. **Problem 15.** A certain percentage of a sphere is cut away. The initial sphere has radius 10, and the volume of the remainder of the sphere is 100π. What percentage of the original sphere remains?

9.16. **Problem 16.** What is the radius of a sphere with the same surface area and volume? Note: the units of the surface area and volume will be different, we just want the numerical value to be the same. In other words, ignore units for this problem.

9.17. **Problem 17.** If a sphere has volume 36π, find its radius.

9.18. **Problem 18.** a. A rectangular prism with a square base has side length s of that square base, and height h. A pyramid sits inside the prism with a square base, side length s, and height h. Find the percentage of the volume of the prism taken up by the pyramid.

b. Now suppose that the pyramid's square base has side length $\dfrac{s}{2}$, and find the percentage of the original prism's volume taken up by the new pyramid.

9.19. **Problem 19.** a. Find the volume of a sphere with diameter 6 centimeters.

b. If we only have $\frac{1}{6}$ of the sphere, what is the new volume?

9.20. **Problem 20.** Determine the volume of a Christmas tree. The top is a pyramid with square base, where the square base has side length 20 inches, and the height is 48 inches. The stand for the tree is a rectangular prism with length 6 inches, width 10 inches, and height 24 inches. Include units in your answer.

10. Graph Theory, Topology, and Curvature

10.1. **Problem 1.** Consider Slide 11 for the Day 8 Story. Pick a different geographical region of the United States and create a similar graph describing which states border each other.

10.2. **Problem 2.** Create another squiggle like the one on Slide 22 of the Day 8 Story. Show that it satisfies Euler's Characteristic Theorem.

10.3. **Problem 3.** If a graph has 12 vertices and 15 faces, how many edges must it have?

10.4. **Problem 4.** Explain why the edges of a square splits the two-dimensional plane into two regions (an inside and an outside).

10.5. **Problem 5.** Two people are lost in a labyrinth and manage to count that there are 17 edges between them. Are they on the same side of the labyrinth?

10.6. **Problem 6.** True or false: on a curved surface, the sum of a triangle's interior angles can be more than 180°

10.7. **Problem 7.** Draw a graph with an Euler path and indicate where that path is.

10.8. **Problem 8.** Sketch a three-dimensional figure that has genus 1.

10.9. **Problem 9.** My travel itinerary for the Thanksgiving holiday in 2021 is as follows: fly from Knoxville to Atlanta, then Atlanta to Pittsburgh. At the end of the holiday, drive from Pittsburgh to Labtrobe, then fly from Latrobe to Fort Lauderdale, then Fort Lauderdale to Knoxville.

a) Draw a graph representing my travel itinerary.

b) Give an example of an Euler circuit on the graph that you drew.

10.10. **Problem 10.** a. Explain why a sphere and a torus are not topologically equivalent.

b. Draw a shape with genus 2.

10.11. **Problem 11.** Draw a labyrinth Jordan Curve and two dots that are on opposite sides of the labyrinth.

10.12. **Problem 12.** Make a table indicating the different types of curvature. In it, distinguish between:

- Intrinsic vs. extrinsic curvature
- Positive, negative, and zero curvature

This table will make for a useful study resource once it's complete.

11. PROBABILITY

11.1. **Problem 1.** Who was responsible for writing *Principles of Games and Chance*?

11.2. **Problem 2.** Explain the law of large numbers in your own words. Note: see DECISION Day 1 Slide 6 for an "official" description.

11.3. **Problem 3.** If you rolled an 8-sided die with faces numbered $1, 2, 3, 4, 5, 6, 7, 8$, all equally weighted, what do you "expect" the average would be after rolling the die many times? (NOTE: by "equally weighted," I mean the probability of landing on each side is $\frac{1}{8}$, since there are 8 sides in total)

11.4. **Problem 4.** Give an example of where the balance fallacy might come into play in your major or discipline (see DECISION Day 1 Slide 19 for a discussion of what the balance fallacy is).

11.5. **Problem 5.** Which of the following is/are an example(s) of the balance fallacy?

A. One of two teams in a soccer game kicks the ball out of bounds, and you assume that there is a 50 percent chance it was the blue team that did it.

B. One of two teams in a soccer game kicks the ball out of bounds, and you conclude one of the two teams should be penalized.

C. One of two teams in a soccer game kicks the ball out of bounds, and you assume that there is a 50 percent chance it was the red team that did it.

D. You flip a fair coin many times and observe the Law of Large Numbers taking place.

11.6. **Problem 6.** If rolling a die many times, why does the average number that shows up on the die *not* need to equal one of the numbers on the die?

11.7. **Problem 7.** You have two square-shaped dartboards, where the squares both have side length 10 cm. Each dartboard has a red circle inside the square. The left dartboard has a circle of radius 2 cm on it. The right dartboard has a circle of radius 3 cm on it.

a. If you throw darts at each dartboard, what is your guess for which red circle will be easier to hit? Explain your reasoning. Assume any dart you throw hits the board you wanted it to.

b. Actually calculate the probabilities of hitting the red circles.

11.8. **Problem 8.** A roulette wheel has 10 green strips, 10 yellow strips, and 10 red strips. Is the probability of landing on a green strip the same as landing on a red strip? Do you even have enough information to answer this question?

12. PERCENTAGES, TAXES, AND DEDUCTIBLES (MARCH 22, MARCH 24, MARCH 26)

12.1. **Problem 1.** If a vase is listed at 60 dollars and there is a 30 percent off sale. How much will you pay for the vase?

12.2. **Problem 2.** If you pay 80 dollars for a new vacuum, with 7 percent sales tax added on at the end, what will the final bill be?

12.3. **Problem 3.** Maybe some of you watch CNBC or read up on the stock market. Many times the behaviors of the stocks are listed as percent changes. Pick one stock whose value changed a certain percentage over a certain period of time, and write a sentence explaining what happened. Include the start and end prices of the stock.

12.4. **Problem 4.** I buy an airplane ticket where the initial cost is 350 dollars. Then, I pay a 30 dollar bag fee and 26 dollars for insurance on the ticket (so I get a refund if the flight is canceled). The extra expenses equal what percent of the original ticket price?

12.5. **Problem 5.** Go to the table in DECISION Day 4, Slide 6. Using the data in that table, find the amount of your income that will be taxed if your annual income is $80,000$ dollars. Then, check your answer using the tax schedule on Slide 10.

12.6. **Problem 6.** If you have a 1700-dollar deductible on auto insurance, then have a car accident, how much money would you need to pay up front before an insurance claim would help cover some of the expenses?

12.7. **Problem 7.** Suppose your medical expenses for the year are 5300 dollars, and your deductible is 2300. If your insurance policy provider pays for 80 percent of the expenses beyond the deductible, how much money will need to come out of your pocket? **Note:** there are actually two ways to do this problem.

12.8. **Problem 8.** Your hospital bill from treatment from a disease comes out to be 6400 dollars. Your medical insurance plan has a 1700 dollar deductible, and then you cover 25 percent of the remaining expenses after that.

a. How much money do you have to pay?

b. How much money does your insurance provider pay?

c. If you increase the size of the deductible but the total hospital bill amount stays the same, will the total amount you pay increase, decrease, or stay the same?

12.9. **Problem 9.** Your bill for treatment of your house from a termite infestation totals to be 5100 dollars. After you pay the deductible, your insurance provider pays 80 percent of remaining costs. The total amount, including the deductible, that you pay is 1500 dollars. What is the size of the deductible?

NOTE: I expect this problem to be a little bit harder since it's not exactly like any we've done in class but give it a try.

12.10. **Problem 10 (student problem).** According to the Census, the population in the United States in 2010 is $309,011,475$. The population in the United States increased to $331,002,651$ in 2020. What is the percent increase in the population from 2010 to 2020?

12.11. **Problem 11 (student problem).** The current American population in the year 2020 is 334.5 million people. The projected population for the year 2030 is 359.4 million people. Given this information, find the percent of the population increase between the year 2020 and 2030. Round to the nearest tenth.

12.12. **Problem 12 (student problem).** The current population of the US in the year 2020 is 334.5 million, and the estimated population for the year 2040 is 380.22 million. Given this information, what is the estimated percent increase of the country's population between the year 2020 and 2040?

12.13. **Problem 13 (student problem).** In 2019, the government received 3.46 trillion dollars in income tax. In the case of raising taxes by 25 percent, what would you estimate the new total to be?
A. 4.965 trillion dollars
B. 3.809 trillion dollars
C. 5.143 trillion dollars
D. 4.325 trillion dollars

12.14. **Problem 14 (student problem).** Which of the following is the correct formula for percent *increase*?
A. $\dfrac{\text{final} - \text{initial}}{\text{initial}} * 100\%$
B. $\dfrac{\text{initial} - \text{final}}{\text{initial}} * 100\&$
C. $\dfrac{\text{final} - \text{initial}}{\text{final}} * 100\%$
D. None of the above

12.15. **Problem 15 (student problem).** Suppose a city's total power usage in 2010 was $15,372,000$ mWh (Megawatt-hours). If the total power usage increased to $17,251,000$ mWh in 2020, what is the percent increase in total energy used from 2010 to 2020? Round your answer to the nearest tenth of a percent.

12.16. **Problem 16 (Finance Problem).** On January 4, 2021 the share price of Apple stock (AAPL) was 129.41 dollars. On March 4, 2021, the share price was 120.13 dollars. What is the percent decrease of the stock price in this time period?

For the next three problems, use the tax table provided.

A.G.I. (adjusted gross income)	Taxes Owed
\$0 to \$9,875	10% of income
\$9,876 to \$40,125	\$987.50, plus 12% of the amount beyond \$9,875
\$40,126 to \$85,525	\$4,617.50, plus 22% of the amount beyond \$40,125
\$85,526 to \$163,300	\$14,605.50, plus 24% of the amount beyond \$85,525
\$163,301 to \$207,350	\$33,271.50, plus 32% of the amount beyond \$163,300
\$207,351 to \$518,400	\$47,367.50, plus 35% of the amount beyond \$207,350
Over \$518,400	\$162,456, plus 37% of the amount beyond \$518,400

12.17. **Problem 17.** Your friend remembers that she owes $\$14,000$ in taxes, as 14 is her favorite number, a fun coincidence. Using the tax schedule, what is her adjusted gross income (A.G.I.)? **Note: this problem is from DECISION: Day 4**

12.18. **Problem 18.** If you made $\$66,250$ in income this year:
a. How much money will you owe in taxes?
b. How much of your income will you get to keep? Express your answer as a percentage.

12.19. **Problem 19.** a. What are the smallest and largest possible amounts of taxes owed if you are looking at the third row in the table?
b. If your adjusted gross income is $\$900,000$, how much money is owed in taxes?

13. Expected Value

13.1. **Problem 1.** If there is a 70 percent chance of rain every day, how many rainy days do you expect in a two-week period? That is, find expected value.

13.2. **Problem 2.** If any person has a 40 percent chance of deciding to get a flu vaccine, and a vaccine costs 30 dollars, what is the expected amount a person spends on a flu vaccine? Note: I do not guarantee these numbers to be realistic.

13.3. **Problem 3.** If a flu vaccine costs 30 dollars, and the expected amount a person spends on a flu vaccine is 19.50 dollars, what is the probability that a random person gets a flu vaccine? Note: I do not guarantee these numbers to be realistic.

13.4. **Problem 4.** If any person has a 70 percent chance of deciding to get a flu vaccine, and a vaccine costs 40 dollars, what is the expected amount a person spends on a flu vaccine? Note: I do not guarantee these numbers to be realistic.

13.5. **Problem 5 (Finance Problem).** Your oil stock is currently priced at 30 dollars per share, and you own 10 shares. In the next month, there is a 20 percent chance the price will go up to 45 dollars, a 20 percent chance the price will go down to 25 dollars, and a 60 percent chance the price will remain unchanged. What is the expected value of your shares at the end of the month?

14. Interest

14.1. **Problem 1.** An engagement ring is pawned at a pawn shop for 500 dollars (yes this is a pitiful rate) and the interest rate the pawn shop charges is 12 percent per month (simple interest). How much money must you repay the pawn shop if you wait:

a. 2 months after pawning the ring

b. 4 months after pawning the ring

c. If you were to still pawn the ring at 500 dollars, wait 6 months, and then owe 1000 in total, what would the new monthly interest rate be?

14.2. **Problem 2.** Take the equation $I = P \cdot r \cdot t$ from Slide 8 of DECISION Day 7 Story, and rearrange it to be an equation solving for r. That is, put r on one side of the equation, and everything else on the other side.

14.3. **Problem 3.** The principal value of an investment is 3000 dollars. The simple interest rate is 6 percent per month, over 8 months. How much interest accumulates in those 8 months? Give your answer as a dollar amount.

14.4. **Problem 4.** You invest 700 dollars in the bank for 6 months at an annual simple interest rate of 5 percent.

a. What is the amount of interest that accumulates in this time period?

b. If you withdraw all the money at the end of 6 months, how much money will you be withdrawing?

14.5. **Problem 5.** Explain in words why compound interest grows faster than simple interest.

14.6. **Problem 6.** If an investment has initial value 1200 dollars and 5 percent interest is compounded 10 times per year, what will the total value of the investment be after:

a. 1 year

b. 2 years

14.7. **Problem 7.** Explain how a pawn shop works through a short essay response. Answer each of the following questions in 1 to 2 sentences each:

a. Explain the initial transaction that takes place (who gets what)

b. How does the pawn shop profit off of an item that the owner does redeem?

c. How does the pawn shop profit off of an item that the owner does *not* redeem?

14.8. **Problem 8.** You invest P dollars in a bank and earn 22 percent interest compounded quarterly (yikes), over a time period of 5 years. There is now 2000 dollars in the account. How much money was in the account to start with?

15. Voting Methods

15.1. **Problem 1.** Which of the following fairness attributes is described by, “If one wins, and then if there is a re-election, if all changes favor that one, then that one should still win."
A. Monotonicity
B. Majority
C. Condorcet
D. Independence

15.2. **Problem 2.** Suppose three candidates are running in an election, and each person voting lists their first choice, second choice, and third choice. In the plurality method, which of these is the only one that is taken into account?
A. First choices
B. Second choices
C. Third choices
D. They are all taken into account

15.3. **Problem 3.** Do you think plurality with elimination is more fair than plurality without elimination? Explain why or why not.

15.4. **Problem 4.** Which of the following is best described by "each candidate gets points for each ranking, and more points for higher rankings?"
A. Plurality
B. Plurality with elimination
C. Borda count
D. Arrow’s impossibility theorem

15.5. **Problem 5.** a. If you conduct an election using pairwise comparison with three different parties, how many pairs would you need to compare?
b. If you conduct an election using pairwise comparison with four different parties, how many pairs would you need to compare?

15.6. **Problem 6.** Go back to the car, boat, train table on Slide 14 of the DECISION Day 10 slides. The first row of the table has the number of people who voted for each preference order. Using some trial and error, change the numbers in the top row in such a way that the train wins by Borda Count instead of car. (NOTE: this problem will likely take a while compared to others but it will give you practice using the Borda count on a preference table)

15.7. **Problem 7 (student problem).** When voting using the plurality method, the candidate with the second most votes wins. True or false?

15.8. **Problem 8 (student problem).** The Borda Count determines the winner from all the candidates based on having the least number of votes. True or false?

15.9. **Problem 8 (student problem).** Explain in words what the Borda count is and how it works.

15.10. **Problem 9.** A group of four people ranked their favorite meal from best to worst in the following way:

1 person ranked lunch, dinner, breakfast
2 people ranked breakfast, dinner, lunch
1 person ranked dinner, lunch, breakfast
a. Which meal wins with plurality?
b. What happens if you try to find a winner with plurality with elimination?

15.11. **Problem 10 (this problem was inspired by a discussion with a student).** If you have a vote between three choices, and 60 voters participate, ranking their preferences first, second, and third, what is the total number of points given to all the preferences using Borda Count?

16. Answers

Here are some answers to problems.

Problem 2.1: $S_n = n^2$ is non-recursive; $S_{n+1} = S_n + 2n + 1$ is recursive

Problem 2.2: To get the next term in the sequence, multiply the previous term of the sequence by 3 and add 1. A recursive formula is $S_{n+1} = 3S_n + 1$.

Problem 2.3: The recursive formula does not satisfy the non-recursive formula.

Problem 2.4: The recursive formula does not satisfy the non-recursive formula.

Problem 2.5: To get the next term in the sequence, multiply the previous term of the sequence by 2 and add 1. The next two terms will be 95 and 191.

Problem 2.8: 14

Problem 2.11: The numbers generated are 1 less than the powers of 2.

Problem 2.12: Take the previous number, multiply it by 2, then subtract 2. Possible recursive formulas are: $R_n = 2R_{n-1} - 2$ and $R_n = R_{n-1} + 5 \cdot 2^{n-1}$.

Problem 3.1: True

Problem 3.2: False. Consider $0, -1, -2, -3, \ldots$

Problem 3.3: True

Problem 3.4: $\sqrt{2}, \pi, \sqrt{3}, e, \sqrt{5}$

Problem 3.6: Pair off 1 with 2, 2 with 4 , 3 with 6 , and so on.

Problem 3.7: Results may vary

Problem 3.8: This set is finite because there are 26 choices for the first letter, and 26 for the second, so there are $26 \times 26 = 676$ words, which is finite.

Problem 3.9: This conjecture is true! A prime number has two factors, one and itself. When checking if a number is perfect, we sum all of its factors except for the number itself. If we do this test on a prime number, the sum of all the factors we care about will just be 1. But every prime number is bigger than 1, so no prime number is perfect!

Problem 4.1:

a. The size of [set of] the prime numbers is the same as the size of the [set of] natural numbers

b. Since the natural numbers are $\aleph_0$ and the size of [set of] the prime numbers is the same as the size of the [set of] natural numbers, the set of prime numbers are $\aleph_0$

Problem 4.2:

a. 4 (the only prime factor is 2)

b. 15 (the only prime factors are 3 and 5)
Problem 4.3:
a. $4 = 2 + 2$; this is the only way; we've built a sum using the same prime number twice.
b. $22 = 5 + 17 = 3 + 19 = 11 + 11$
c. $94 = 47 + 47$
Problem 4.4:
a. No; the remainder is 1
b. Yes; the remainder is 0
c. The remainder is 1 for any of the divisions.
Problem 4.5:
3 and 7; 7 and 11 (there are many other pairs)
Problem 4.7: LCM is 72; GCD is 12
Problem 4.8: LCM is 480; GCD is 4
Problem 4.9: results may vary
Problem 4.10: B
Problem 5.1: a. 7
b. $F_6 = 8$
c. $F_{11} = 89$
Problem 5.2:

$$H_3 = 5, H_4 = 9, H_5 = 14, H_6 = 23$$

Also, $H_9 - H_7 = H_8 = 60$.

Problem 5.3: Every sixth Fibonacci number is divisible by 4; every fourth Fibonacci number is divisible by 3

Problem 5.4: $2\psi + \psi^2 = 2$

Problem 5.5: $\sqrt{5}$ is irrational. Adding 1 and dividing by 2 doesn't change that

Problem 5.6: False. All of these numbers are rational, because they are fractions between two integers.

Problem 6.1: We just took the reciprocal of both sides of the equation on the slide.

Problem 6.2: The simpler model is probably less accurate.

Problem 6.3: Exponential models grow faster over time; the shape of the line going through (or near) the data points is curved for an exponential model, and a straight line for the linear model; other answers possible.

Problem 6.6: True
Problem 6.7: 48.6 meters
Problem 6.8: C
Problem 6.9: 34.3 (this is rounded to nearest tenth)
Problem 6.10: 2.46 feet
Problem 6.11: True
Problem 6.12: 472 pounds
Problem 6.13: E
Problem 6.14: C

Problem 6.15: 2869.57 pounds
Problem 7.1: 50 degrees
Problem 7.2: 85 degrees
Problem 7.3: $45^2 = 2025$
Problem 7.4: Results vary greatly. You should just make sure you get the same angle measures for A, B, C, D, E as we did in class.
Problem 7.5: No. If two angles each have measure 90 degrees, and the sum of all three angles is 180 degrees, the third angle would have measure 0 degrees, which is impossible.
Problem 7.6: N/A
Problem 7.7: 4 or $\sqrt{34}$
Problem 7.8: 10
Problem 7.9: No. A right triangle would have to satisfy the Pythagorean Theorem, $4^2+7^2 = 9^2$. This is not actually true.
Problem 7.10: a. $EF = 18$ meters
b. No we need more information to find AC or DF. If you knew either of these side lengths you would be able to find the other.
Problem 8.1: 16π square cm
Problem 8.2: 25π square cm
Problem 8.3: 6 cm
Problem 8.4: $144\pi - 4$ square cm (or 448.39 square cm)
Problem 8.5: 1.56
Problem 9.1: Surface area: 150; volume 125
Problem 9.2: Surface area: 146; volume 90
Problem 9.3: The side length of the cube would have to be 6.
Problem 9.4: a. 600π square feet
b. 2000π cubic feet
Problem 9.5: Answers may vary
Problem 9.6: Answers may vary
Problem 9.7: 1 cubic foot
Problem 9.8: The cube is the combination of three pyramids that, although they have different shapes, have the same volume by Cavalieri's Principle. Therefore we can calculate the volume of the cube and then divide that by 3.
Problem 9.9: Answers may vary
Problem 9.10: 4.531×10^7 square miles
Problem 9.11: 324π
Problem 9.12: 144π
Problem 9.13: 4
Problem 9.14: 144π
Problem 9.15: 7.5 percent
Problem 9.16: 3
Problem 9.17: 3
Problem 9.20: 7840 cubic inches
Problem 10.1: Answers may vary

Problem 10.2: Answers may vary

Problem 10.3: 25

Problem 10.4: A square splits the plane into a region completely enclosed by the sides of the square, and a second region completely separate from this. You can color the inside of the square blue, and the outside orange, for instance.

Problem 10.5: Opposite sides

Problem 10.6: True

Problem 10.7: Answers may vary

Problem 10.8: Answers may vary, but I imagine a torus will be the most common answer

Problem 10.9: a. Drawing not available; b. Knoxville -> Atlanta -> Pittsburgh -> Latrobe -> Fort Lauderdale -> Knoxville

Problem 10.10: a. A sphere has genus 0 and a torus has genus 1; b. Results may vary.

Problem 10.11: Results may vary

Problem 10.12: Answers may vary

Problem 11.1: Christiaan Huygens

Problem 11.2: Answers may vary

Problem 11.3: 4.5

Problem 11.4: Answers may vary. Be creative! This is one question where Google might give you some inspiration.

Problem 11.5: A and C are both correct.

Problem 11.6: The average can be a number in between all the numbers on the die. For instance, if the die has sides labeled $1, 2, 3, 4, 5, 6$, the average that shows up can be a decimal. Decimals are numbers that lie in between the integers on the number line. If the die is fair, the average will be 3.5 in this case.

Problem 11.7:

a. The red circle on the right dartboard will be easier to hit because it is bigger (important to note that the squares have the same size)

b. Left dartboard: $\frac{4\pi}{100} \approx .126$; right dartboard: $\frac{9\pi}{100} \approx .283$

Problem 11.8: We do not have enough information to solve this problem. We don't know whether the strips all have the same size! If they do, then the probability of landing on a green strip equals the probability of landing on a red strip.

Problem 12.1: 42 dollars

Problem 12.2: 85.6 dollars

Problem 12.3: This is an open-ended problem and answers will vary. One simple example: the price of stock X was 100 dollars on January 1, 2019. Between then and June 30, 2019, the stock X price fell 40 percent to 60 dollars.

Problem 12.4: 16 percent

Problem 12.5: 13390 dollars

Problem 12.6: 1700 dollars

Problem 12.7: 2900 dollars

Problem 12.8:

a. 2875 dollars

b. 3525 dollars
c. Increase
Problem 12.9: 600 dollars
Problem 12.10: 7.12 percent
Problem 12.11: 7.4 percent
Problem 12.12: 13.7 percent
Problem 12.13: 4.325 trillion dollars
Problem 12.14: A
Problem 12.15: 12.2 percent
Problem 12.16: 7.2 percent
Problem 12.19:
a. Between $\$4617.50$ and $\$14605.50$
b. $\$303648$
Problem 13.1: 9.8 days
Problem 13.2: 12 dollars
Problem 13.3: 0.65 or 65 percent
Problem 13.4: 28 dollars
Problem 13.5: 320 dollars
Problem 14.1:
a. 620 dollars
b. 740 dollars
c. 16.7 percent (approximately)
Problem 14.2: $r = \dfrac{I}{Pt}$
Problem 14.3: 1440 dollars
Problem 14.4:
a. 17.50 dollars
b. 717.50 dollars
Problem 14.5: Each time compound interest is compounded, the amount is calculated based on the total amount of money, rather than just the principal amount. This is an example of comparing exponential growth to linear growth.
Problem 14.6:
a. 1261.37 dollars
b. 1325.87 dollars
Problem 14.7: Answers may vary
Problem 14.8: $P = 685.64$
Problem 15.1: A
Problem 15.2: A
Problem 15.3: Answers may vary; an argument in favor of plurality with elimination will likely mention that second and third choices get taken into account; an argument in favor of plurality without elimination will likely mention that the greatest number of first choices identifies a winner
Problem 15.4: C
Problem 15.5: a. 3; b. 6

Problem 15.6: Answers may vary
Problem 15.7: False
Problem 15.8: False
Problem 15.9: a. Breakfast; b. There is a tie between two options for the meal with the least number of first-place votes
Problem 15.10: 360