APPROXIMATION THEOREMS: MEASURE THEORY

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1. Introduction

This document is designed as a catalogue of different approximation theorems that appear in measure theory. The broad theme is functions or sets that are approximated in some sense by other functions or sets. The exact sense of approximation varies greatly but nonetheless it seemed suitable to list these results in a compact setting. None of the results were originally proven by the author and are more readily attributed to texts in classical measure theory. Each subsection constitutes a different theme of approximation. Throughout assume m is the Lebesgue measure on $\mathbb{R}$ and μ is some arbitrary positive measure, unless stated otherwise.

2. Approximation of Sets

Theorem 2.1 (Bass p. 29)**.** *Let $A \subset [0,1]$ be Borel measurable, m denote Lebesgue measure.*

i) Given $\epsilon > 0$, there exists open set G such that $m(G - A) < \epsilon$ and $A \subset G$.

ii) Given $\epsilon > 0$, there exists closed set F such that $m(A - F) < \epsilon$ and $A \subset F$.

iii) There exists a set H with $A \subset H$ that is the countable intersection of a decreasing sequence of open sets where $m(H - A) = 0$.

iv) There exists a set F with $F \subset A$ that is the countable union of an increasing sequence of closed sets all contained in A where $m(A - F) = 0$.

3. Functions

Lemma 3.1 (Bass p. 41)**.** *Let f be a nonnegative and measurable function. Then there exists a sequence of non-negative measurable simple functions s_n which increase to f.*

Theorem 3.2 (Bass p. 42)**.** *(Lusin's Theorem) Let $f : [0,1] \to \mathbb{R}$ be Borel measurable and $\epsilon > 0$. Then there exists $F \subset [0,1]$ closed such that $m([0,1] - F) < \epsilon$ and the restriction of f to F is continuous on F.*

Theorem 3.3 (Bass p. 78)**.** *(Egorov Theorem) Let μ be a finite measure, $\epsilon > 0$ and $f_n \to f$ a.e. Then there exists a measurable set A such that $\mu(A) < \epsilon$ and $f_n \to f$ uniformly on A^C.*

The above theorem is colloquially referred to as *almost uniform convergence*.

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4. Integral Convergence Theorems

Theorem 4.1 (Bass p. 50)**.** *(Bounded Convergence Theorem) Suppose $\mu(X) < \infty$ and f_n is a sequence of bounded real-valued measurable functions that converge to f uniformly. Then*

$$\int f_n d\mu \to \int f d\mu.$$

Theorem 4.2 (Bass p. 51)**.** *(Monotone Convergence Theorem) Suppose f_n is a sequence of non-negative measurable functions where $f_1(x) \leq f_2(x) \leq \ldots$ for all x and $\lim_{n\to\infty} f_n(x) = f(x)$ pointwise on X. Then*

$$\int f_n d\mu \to \int f d\mu.$$

Lemma 4.3 (Bass p. 54)**.** *Let f_n be nonnegative measurable functions. Then*

$$\int \sum_{n=1}^{\infty} f_n d\mu = \sum_{n=1}^{\infty} \int f_n d\mu.$$

Lemma 4.4 (Bass p. 55)**.** *(Fatou's Lemma) Let f_n be nonnegative measurable functions. Then*

$$\int \mathit{liminf}_{n\to\infty} f_n d\mu \leq \mathit{liminf}_{n\to\infty} \int f_n.$$

Theorem 4.5 (Bass p. 55)**.** *(Dominated Convergence Theorem) Let f_n be measurable real-valued functions where $f_n \to f$ pointwise on X. Suppose there exists a nonnegative integrable function g such that $|f_n| \leq g$ on X for all $n \in \mathbb{N}^+$. Then*

$$\int f_n d\mu \to \int f d\mu.$$

5. Integral Approximations

Theorem 5.1 (Bass p. 65)**.** *Suppose f is a Borel measurable real-valued integrable function on $\mathbb{R}$, and $\epsilon > 0$. Then there exists a continuous, compactly supported function g for which*

$$\int |f - g| d\mu < \epsilon.$$

6. Completeness

Theorem 6.1 (Bass p. 134)**.** *The space $L^p(\mu)$ is complete for any $1 \leq p \leq \infty$.*

Theorem 6.2 (Bass p. 136)**.** *The set of continuous functions with compact support is dense in $L^p(\mathbb{R})$ for any $1 \leq p \leq \infty$.*

Corollary 6.3 (Bass p. 136)**.** *The set of continuous functions on $[a, b]$ are dense in the space $L^2([a, b])$ with respect to $L^2([a, b])$ norm.*

References

[Ba] R. Bass, "Real Analysis for Graduate Students," 2nd edition, 2013.