Refer on Norms (sec. 19~22) – 4/9/18

1. A norm is a function $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ s.t.
   i) $\|x\| \geq 0 \ \forall x \in \mathbb{R}^n$
   ii) $\|x\| = 0 \iff x = \bar{0}$
   iii) $\|\lambda x\| = |\lambda| \|x\| \quad \forall x \in \mathbb{R}^n, \lambda \in \mathbb{R}$
   iv) $\|x+y\| \leq \|x\| + \|y\| \quad \forall x, y \in \mathbb{R}^n$
2. The 1-norm is a norm on $\mathbb{R}^n$ and is defined as
   $\|\cdot\|_1 : f(x) = \sum_{i=1}^{n} |x_i|$
3. The ∞-norm is a norm on $\mathbb{R}^n$ and is defined as
   $\|\cdot\|_\infty : f(x) = \max_{1 \leq j \leq n} |x_j|$
4. The 2-norm is a norm on $\mathbb{R}^n$ and is defined as
   $\|\cdot\|_2 : f(x) = x^T x = \sum_{i=1}^{n} |x_i|^2 = \langle x, x \rangle$
5. Cauchy-Schwarz: $\forall x, y \in \mathbb{R}^n$,
   $$|x^T y| \leq \|x\|_2 \|y\|_2$$
6. A real inner product is a function $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ where $\forall x, y, z \in \mathbb{R}^n$, $\alpha, \beta \in \mathbb{R}$ are such that
   i) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$
   ii) $\langle x, y \rangle = \langle y, x \rangle$
   iii) $\langle x, x \rangle \geq 0$
   iv) $\langle x, x \rangle = 0 \iff x = \bar{0}$
7. A matrix A is symmetric if $A^T = A$.
8. A symmetric matrix is positive definite if
   $$x^T A x \geq 0 \quad \forall x \in \mathbb{R}^n$$
   and equality holds only if $x = \bar{0}$.
9. If $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ is a norm on vectors, the induced norm of a matrix is
   $$\|A\| := \max_{\|x\|=1} \|Ax\|$$
   and will always be a norm where the domain is $\mathbb{R}^{n \times n}$.

10. If $\|\cdot\|$ is an induced norm on matrices, $\|I\| = 1$.

11. For an induced matrix norm $\|\cdot\|$,

$$\|AB\| \leq \|A\| \cdot \|B\|$$

12. The function $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ via

$$\|x\|_A^2 = x^T A x$$

is a norm if A is symmetric and positive definite.

13. Let λ_j be the eigenvalues of $A^T A$. Then

$$\|A\|_2^2 = \max_j \lambda_j$$

14. If $\|\cdot\|$ is an induced norm on matrices for which $\|A\| < 1$ then $(I - A)^{-1} = \sum_{k=0}^{\infty} A^k$.