OPTIMALITY CONDITIONS FOR FRACTIONAL LAPLACIAN

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CONTENTS

1. GOALS

The goal of this note is to discuss the development of optimality conditions for an optimal control problem where the state equation (serving as a constraint) is the Fractional Laplacian. Here's the setup: we want to minimize the cost functional

$$\min J(u,z) = \frac{1}{2}\|u-u_d\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|u\|^2_{L^2(\Omega)},$$

subject to the state equation constraint

$$\begin{cases} (-\triangle)^s u(x) = z(x), \ x \in \Omega \\ u(x) = 0, \qquad\qquad x \in \mathbb{R}^n \setminus \Omega \end{cases}$$

Our parameters are as follows: $s \in (0,1), \lambda > 0$, and $u_d \in L^2(\Omega)$ are given, and the set of admissible controls is

$$Z_{\text{ad}} := \{\xi \in L^2(\Omega), a \le \xi \le b \text{ a.e.}\}$$

Finally, we note that the Fractional Laplacian $(-\triangle)^s$ is an operator of fractional type defined as

$$(-\triangle)^s u(x) := C(n,s) \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}} dy$$

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2. The State Equation

To have any hope of studying optimality conditions or the optimal control problem as a whole, we must first have well-posedness of the state equation.

Theorem 2.1. *Our state equation, a Dirichlet BVP for the Fractional Laplacian, has a unique solution, and there is a continuous dependence on the data (i.e., a stability estimate).*

Let's consider our function space. We define

$$X_0 := \widetilde{H}^s(\Omega) = \left\{ u \in L^2(\Omega), \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dydx < \infty, u = 0 \text{ on } \mathbb{R}^n \setminus \Omega \right\} :=$$

$$\{u \in L^2(\Omega), [u]_{\widetilde{H}^s(\Omega)} < \infty, u = 0 \text{ on } \mathbb{R}^n \setminus \Omega\}$$

This space is Hilbert with the inner product

$$\langle u, v \rangle_{\widetilde{H}^s(\Omega)} = \langle u, v \rangle_{L^2(\Omega)} + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{n+2s}} dydx$$

and the norm

$$||u||_{\widetilde{H}^s(\Omega)} = \|u\|_{L^2(\Omega)} + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dydx$$

That is, our state equation problem is posed on a Hilbert Space! Weak solutions of the Fractional Laplace Equation will satisfy the following for all $v \in \widetilde{H}^s(\Omega)$ (see for instance [Ser]):

$$C(n,s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{n+2s}} dydx = \int_{\mathbb{R}^n} z(x)v(x)dx,$$

of course with the same zero boundary condition as before. Clearly the left-hand side is a bi-linear form on $\widetilde{H}^s(\Omega) \times \widetilde{H}^s(\Omega)$ and the right-hand side is a linear form on $L^2(\Omega)$ (where z is taken as fixed data and v as a test function). In order to see that we are in a Lax-Milgram framework, we need to check some other properties of the bi-linear form. First, notice we have boundedness (continuity) through Cauchy-Schwarz:

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{n+2s}} dydx \leq \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dxdy \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|v(x) - v(y)|^2}{|x-y|^{n+2s}} dxdy \right)$$

The other thing to remark is that coercivity is immediate because of the equality

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dxdy = [u]^2_{H^s(\Omega)}$$

Therefore, by the Lax-Milgram Theorem (see for instance Corollary 5.8 in [Bre]), there is a unique weak solution $u \in \widetilde{H}^s(\Omega)$ to the Fractional Laplace Equation for any given data $z \in L^2(\Omega)$.

As for continuous dependence, set $v := u$ in the weak formulation:

$$C(n,s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}dxdy=\int_{\mathbb{R}^n}z(x)u(x)dx$$

Use the Cauchy-Schwarz Inequality on the RHS to conclude there exists a constant $C > 0$ (different from above) such that

$$[u]^2_{H^s(\Omega)} \leq C\|z\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}$$

To proceed, we need the Fractional Poincaré Inequality

$$\|u\|_{L^2(\Omega)} \leq C[u]_{H^s(\Omega)}$$

We use it on the greater side of our to-be stability estimate to get

$$[u]^2_{H^s(\Omega)} \leq C\|z\|_{L^2(\Omega)}[u]_{H^s(\Omega)} \Rightarrow$$
$$[u]_{H^s(\Omega)} \leq C\|z\|_{L^2(\Omega)},$$

which is certainly a stability estimate. We hence have well-posedness for our state equation.

3. Reduced Cost Approach

Henceforth let $S : L^2(\Omega) \to W^{s,2}(\Omega)$ be the control-to-state mapping. Then due to the well-posedness of the state equation, solving the original optimal control problem is equivalent to solving the **reduced problem**

$$\min\left\{F(z) := \frac{1}{2}\|Sz-u_d\|^2_{L^2(\Omega)}+\frac{\lambda}{2}\|z\|^2_{L^2(\Omega)}\right\}$$

over $z \in Z_{\text{ad}}$, subject to the same constraint. Notice that our set Z_{ad} is non-empty, closed, convex, and bounded. Since the solution mapping S is continuous, F is continuous; since S is linear, F is convex. In fact, since $\lambda > 0$, F is strictly convex. With this mind we have a unique minimizer for the reduced problem over our desired domain, and in turn we have a unique minimizer for the original [constrained] optimal control problem. We have implicitly used the following theorem and corollary to reach this conclusion.

Theorem 3.1 (Projection Theorem)**.** *Let $(Z, \|\cdot\|)$ be a Hilbert Space, $K \subset Z$ be closed and convex, and let $\xi \in Z$. Then there exists a projection onto K (i.e. a $z \in K$) so that*

$$\|\xi - z\| = \inf_{x\in K}\|\xi - x\|$$

Moreover, z can be characterized by the variational inequality

$$\langle \xi - z, x - z\rangle \geq 0$$

for all $x \in K$.

Corollary 3.2 (Direct Method as Projection)**.** *Let $F : Z \to \mathbb{R}$ be convex and lower semi-continuous with K as before. Then $\exists\xi \in K$ so $F(\xi) \leq F(x)$ for all $x \in K$. If F is strictly convex, then the choice of ξ is unique.*

4. The Formal Lagrangian Approach

The calculations in this section are formal

We define the Lagrangian to be

$$L(u,z,p) = J(u,z) - \int_\Omega p(-\triangle^s u - z)dx$$

As always, a stationary point will satisfy the conditions $D_uL = 0$, $D_zL \geq 0$, and $D_pL = 0$ in the [linear] operator sense. From these we have the following relations:

$$\langle D_zL, z - h_z\rangle = \langle \lambda z, z - h_z\rangle + \langle p, z - h_z\rangle \geq 0 \Rightarrow \langle z - \left(-\frac{1}{\lambda}\right)p, z - h_z\rangle \geq 0$$
$$D_uL = 0 \Rightarrow \langle u - u_d, h_u\rangle - \langle \triangle^s p, h_u\rangle = 0$$
$$D_pL = 0 \Rightarrow \langle -\triangle^s u - z, h_p\rangle = 0$$

In fact, the equalities tell us that necessary conditions for attaining a stationary point of the Lagrangian are

$$-\triangle^s p = u - u_d$$
$$-\triangle^s u = z$$

And, based on our projection characterization, $z = \mathbb{P}_{Z_{\text{ad}}}\left(-\frac{1}{\lambda}p\right)$. For our particular choice of set Z_{ad}, $z = \max\left\{a, \min\left\{b, -\frac{1}{\lambda}p\right\}\right\}$. One advantage of this particular choice of Z_{ad} is that the projection formula is nice and easy to write concisely.

5. The Lagrangian Approach

We need to directly calculate criteria for which

$$\langle DF(z), h_z - z\rangle \geq 0$$

for all $h_z \in Z_{\text{ad}}$. To do this, we set up a difference quotient:

$$F(z+th)-F(z) = \frac{1}{2}\|S(z+t(h-z))-u_d\|^2_{L^2(\Omega)} - \frac{1}{2}\|Sz-u_d\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|z+th\|^2_{L^2(\Omega)} - \frac{\lambda}{2}\|z\|^2_{L^2(\Omega)} =$$
$$\frac{t}{2}\langle S(h-z), zSz - u_d\rangle + \frac{t^2}{2}\|S(h-z)\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|z+th\|^2_{L^2(\Omega)} - \frac{\lambda}{2}\|z\|^2_{L^2(\Omega)} =$$
$$t\langle S^*(Sz-u_d), h-z\rangle + O(t^2) + \frac{\lambda}{2}\|z+th\|^2_{L^2(\Omega)} - \frac{\lambda}{2}\|z\|^2_{L^2(\Omega)}$$

These quantities are all non-negative so long as z refers to a solution of the optimal control problem.

To actually obtain a derivative, we have to divide across by t and then send $t \to 0^+$. The latter two terms simply become $\lambda\langle z, h\rangle$ owing, for instance, to the example presented in Section 2.6 of [Trö]. As a result, we can combine the two terms and get the inequality

$$\langle S^*(Sz - u_d) + \lambda z, h_z - z\rangle \geq 0$$

In fact, the converse holds: that is, if the above inequality holds for all $h_z \in Z_{\text{ad}}$, then z is a solution to the optimal control. The convexity of the cost functional is crucial for this realization.

Remark 5.1. *The development of the variational inequality follows the statements and proofs of Lemma 2.21 and Theorem 2.22 in* [Trö].

Now, provided that $S \in B(L^2(\Omega))$, $p = S^*(Sz - u_d) \in L^2(\Omega)$, then it follows that

$$\left\langle z - \left(-\frac{1}{\lambda}p\right), z - h_z\right\rangle \geq 0$$

for all $z \in Z_{\text{ad}}$.

Thus by reduction, the task that remains is to characterize the adjoint. Let $u := Sz \in H_0^1(\Omega)$, then we want $\langle Sz, p\rangle = \langle z, S^*\varphi\rangle \; \forall z \in L^2(\Omega)$, and $\langle Sz, p\rangle = \langle u, \varphi\rangle$, where $\varphi = u - u_d$.

Now we claim that $p = S^*(Sz - u_d)$ for all $z \in Z_{\text{ad}}$ (where u was chosen to correspond to an arbitrary z via S). To see this, notice that since $u = Sz$, we have that for all $w \in H_0^s(\Omega)$,

$$\langle z, w\rangle = \langle u, w\rangle \Rightarrow \int_\Omega zw dx = C(n,s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(u(x)-u(y))(w(x)-w(y))}{|x-y|^{n+2s}}dxdy$$

so now, we set $z := u - u_d$ and $w := p$, and we obtain

$$\int_\Omega p(u-u_d)dx = C(n,s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(u(x)-u(y))(p(x)-p(y))}{|x-y|^{n+2s}}dxdy \Rightarrow$$
$$\int_\Omega p(Sz-u_d)dx = C(n,s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(u(x)-u(y))(p(x)-p(y))}{|x-y|^{n+2s}}dxdy \Rightarrow$$

Since p satisfies the adjoint state equation $C(n,s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(v(x)-v(y))(p(x)-p(y))}{|x-y|^{n+2s}}dxdy = \int_\Omega (u-u_d)v dx$ over $v \in H^s(\Omega)$ by definition of the adjoint state equation (setting $v := u$)[1], we have

$$\langle \varphi, Sz\rangle = \langle Sz, p\rangle \Rightarrow$$
$$\langle S^*\varphi, z\rangle = \langle p, Sz\rangle$$

for all $z \in L^2(\Omega)$, and so $p = S^*\varphi$ (as they agree in the weak sense). In summary, the optimality conditions are the following: for $u \in H_0^s(\Omega)$, the following hold for all $v, w \in H_0^s(\Omega)$:

$$C(n,s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}dxdy = \int_\Omega zv dx$$
$$C(n,s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{(w(x)-w(y))(p(x)-p(y))}{|x-y|^{n+2s}}dxdy = \int_\Omega (u-u_d)w dx$$

[1] also see Definition 3.4 in [D'Elia]

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