An Optimal Control Problem in Peridynamics

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Outline

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Motivation and Origins

Nonlocal equations [or systems] often take the form

$$\begin{cases} Lu &= f, x \in \Omega \\ u &= 0, x \in \Omega_\delta \setminus \Omega \end{cases}$$

For our problem the nonlocal operator is

$$Lu(x) = \frac{1}{2} \int_{\Omega_\delta} H(x,y) \frac{k_\delta(x-y)}{|x-y|^2} Du(x,y) dy$$

Common in solid state mechanics, including peridynamics

Motivation and Origins (continued)

Definition (PD)

Peridynamics (PD) is a nonlocal model for elasticity of solids that uses integrals over derivatives, attributed to Stewart A. Silling

Features:

- Exchanges derivatives in continuum models for integrals (helps address crack formation)
- Treats particles as having a bond between them
- Range of interaction parameterized by δ, called **horizon**
- Material parameters represented by $h(x)$ (e.g., density)

Problem Statement

Find $(\bar{u}, \bar{g}) \in (u_0 + X_0) \times L^2$ such that

$$I_\delta(\bar{u}, \bar{g}) = \min_{g \in L^2(\Omega;\mathbb{R}^n), u \in u_0 + X_0(\Omega;\mathbb{R}^n)} \int_\Omega F(x, u(x))dx + \frac{\lambda}{2}\|g\|^2_{L^2(\Omega;\mathbb{R}^n)}$$

and u and g satisfy

$$B_h(u, w) = \int_\Omega g(x) \cdot w(x), \; w \in X_0.$$

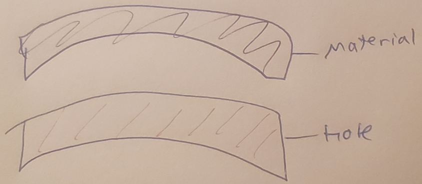
Here $\bar{g}$ is a external force and $\bar{u}$ represents displacement

Candidate integrand

Example

$$F(x, u(x)) = (u(x) - u_{des}(x))^2$$

where U_{des} is the optimal "shaping" of the material in space to fit in some hole.



Goals

- Prove that solutions to the constraint equation exist and are unique
- Identify conditions on F where minimizers exist and are unique
- Consider behavior as $\delta \to 0^+$

Outline

Notation and Properties of Kernels

- Let $\Omega \subset \mathbb{R}^n$ be a bounded domain
- Projected difference: $Du(x,y) := \frac{(u(x)-u(y))\cdot(x-y)}{|x-y|}$, nonlocal linearized strain (for vector-valued functions)
- Kernel sequence $\{k_\delta\}_{\delta>0}$ radial, integrable, non-negative, supported in $B(0,\delta)$, $k_\delta(r)r^{-2}$ is nonincreasing

Properties of Kernel

The following also hold for all $\delta > 0$:

$$\int_{\mathbb{R}^n} k_\delta(\xi)d\xi = 1$$

$$\lim_{\delta \to 0^+} \int_{\mathbb{R}^n} k_\delta(\xi)d\xi = \delta_0$$

$$\lim_{\epsilon \to \infty} \int_{\mathbb{R}^n \setminus B(0,\epsilon)} k_\delta(\xi)d\xi = 0$$

Problem Statement (local part)

Local objective functional:

$$I_0(g,u) \; = \; \int_\Omega F(x,u(x))dx + \frac{\lambda}{2}\|g\|^2_{L^2(\Omega;\mathbb{R}^n)}$$

Constraint: $u \in W^{1,2}(\Omega;\mathbb{R}^n)$ satisfying

$$\begin{cases} b_h(u,w) \; = \; \int_\Omega g(x)\cdot w(x)dx, \;\; w \in W_0^{1,2}(\Omega;\mathbb{R}^n) \\ \qquad\qquad u \; = \; u_0 \text{ in } \partial\Omega \end{cases} .$$

Problem Statement (local part)

Local inner product:

$$b_h(u,v) := C(n)\int_\Omega h(x)(2\langle \mathrm{Sym}(\triangledown u), \mathrm{Sym}(\triangledown v)\rangle_F + \mathrm{div}(u)\mathrm{div}(v))dx,$$

with $C(n) = \frac{1}{(n+2)(n+4)}$; admissible class of pairs for the local problem:

$$\mathcal{A}^{\mathrm{loc}} := \{(v,f) \in W^{1,2}(\Omega;\mathbb{R}^n) \times L^2(\Omega;\mathbb{R}^n), \quad v \text{ solves local BVP for any } w \in W_0^{1,2}(\Omega;\mathbb{R}^n)\}$$

Design Function

Our design function is

$$H(x,y) \;:=\; \frac{h(x)+h(y)}{2},$$

where there exist $h_{\min}, h_{\max} > 0$ so $h_{\min} \leq h \leq h_{\max}$ on Ω.

Symmetry of k_δ allows us to use this representation

Bi-linear Form

For fixed $\delta > 0$:

$$B(u,v) := \int_{\Omega_\delta}\int_{\Omega_\delta} k_\delta(x-y)\frac{Du(x,y)Dv(x,y)}{|x-y|^2}dxdy$$

$$B_h(u,v) := \int_{\Omega_\delta}\int_{\Omega_\delta} H(x,y)k_\delta(x-y)\frac{Du(x,y)Dv(x,y)}{|x-y|^2}dxdy$$

Function Spaces

Our function space is based on the forms B and B_h:

$$X(\Omega_\delta;\mathbb{R}^n) := \{u \in L^2(\Omega_\delta;\mathbb{R}^n), B(u,u) < \infty\}$$

$$\partial X := \{w|_{\Omega_\delta \setminus \Omega}, w \in X\}$$

$$X_0(\Omega_\delta;\mathbb{R}^n) := \{u \in X, u = 0 \text{ in } \Omega_\delta \setminus \Omega\}$$

For $u_0 \in \partial X$, we define the translation

$$u_0 + X_0 \;=\; \{v \in X, v|_{\Omega_\delta \setminus \Omega} = u_0\}$$

Hilbert Space

Lemma

The space $X(\Omega;\mathbb{R}^n)$ *equipped with the norm*

$$\|u\|_{X(\Omega;\mathbb{R}^n)} := \|u\|_{L^2(\Omega;\mathbb{R}^n)} + [u]_{X(\Omega;\mathbb{R}^n)}$$

is a Hilbert Space, and so ix X_0*; here* $[u]_{X(\Omega;\mathbb{R}^n)} = B(u,u)^{\frac{1}{2}}$

Hilbert Space (continued)

For completeness, let $\{u_k\}_{k=1}^{\infty} \subset X(\Omega;\mathbb{R}^n)$ be Cauchy. Then $L^2(\Omega;\mathbb{R}^n)$ gives a candidate limit u. On a sub-sequence,

$$\lim_{m\to\infty} k_\delta(x-y)\frac{|Du_{k_m}(x,y)|^2}{|x-y|^2} = k_\delta(x-y)\frac{|Du(x,y)|^2}{|x-y|^2}$$

Then by Fatou's Lemma,

$$[u]_{X(\Omega;\mathbb{R}^n)} \leq \operatorname{liminf}_{k\to\infty}[u_k]_{X(\Omega;\mathbb{R}^n)} < \infty$$

Hilbert Space (continued)

Finally want $\lim_{k\to\infty}[u_k - u]_{X(\Omega;\mathbb{R}^n)} = 0$. Use that $\{u_k\}_{k=1}^{\infty}$ is Cauchy to see that for $j \in \mathbb{N}^+$ sufficiently large,

$$[u_k - u_j]_{X(\Omega;\mathbb{R}^n)} \ < \ \epsilon.$$

Then use Fatou's Lemma over j once more to conclude completeness.
Also, $B_h(\cdot,\cdot)$ is an inner product only on X_0

Extension Lemma on X_0

Lemma (Extension Lemma)

If $u \in X_0(\Omega_\delta; \mathbb{R}^n)$ and $\widetilde{u}$ is the zero extension to $\mathbb{R}^n$ then there exists a constant $C = C(\delta) > 0$ such that

$$[\widetilde{u}]_{X(B;\mathbb{R}^n)} \;\leq\; C\|u\|_{X(\Omega_\delta;\mathbb{R}^n)}.$$

whenever B is an open set containing Ω_δ; in particular, the constant is independent of B, and we may select $B := \mathbb{R}^n$.

The restriction to X_0 (instead of X) is crucial!

Poincaré Inequality

Theorem (Poincaré)

(Mengesha-Du 2014) There exists a $\delta_0 > 0$ *and a constant* $C(\delta_0) > 0$ *such that for all* $\delta \in (0, \delta_0]$ *and* $u \in L^2(\Omega_\delta; \mathbb{R}^n)$ *vanishing on* $\Omega_\delta \setminus \Omega$*,*

$$\|u\|^2_{L^2(\Omega;\mathbb{R}^n)} \leq C(\delta_0) \int_\Omega \int_\Omega \frac{k_\delta(x-y)|Du(x,y)|^2}{|x-y|^2} dxdy.$$

The assumption of $k_\delta(r)r^{-2}$ being nonincreasing is needed here

Outline

Existence-Uniqueness Result

Theorem (Existence and Uniqueness)

For any $u_0 \in \partial X$ and $g \in L^2(\Omega; \mathbb{R}^n)$, there exists a unique $u \in u_0 + X_0$ such that the state system is satisfied for all $w \in X_0$. Furthermore, we have the stability estimate

$$\|u\|_X \;\leq\; C(\|\widetilde{u}\|_X + \|g\|_{X^*})$$

for some $C > 0$ independent of δ, where $\widetilde{u}$ is an extension of u_0 to all of Ω_δ.

- Start with $u_0 = 0$
- Invoke Riesz Representation Theorem
- Prove stability estimate

The solution here is a variational solution!

Riesz Argument ($u_0 = 0$)

Suppose $u_0 = 0$. Notice that X_0 (and X) is Hilbert with the norm

$$\|u\|_H^2 := \|u\|^2_{L^2(\Omega;\mathbb{R}^n)} + B_h(u,u)$$

and this norm is equivalent to $\|\cdot\|_X$. By nonlocal Poincaré,

$$c\|u\|_H^2 \leq B_h(u,u) \leq \|u\|_H^2.$$

Since $B_h(\cdot,\cdot)$ is an inner product on (X_0, B_h), we may use Riesz to uniquely find u satisfying

$$B_h(u,w) = \int_\Omega w(x)\cdot g(x), \ \forall w \in X_0$$

for each $g \in X_0^*$.

Riesz Argument ($u_0 \neq 0$)

If $\widetilde{u}$ extends u_0 from Ω to Ω_δ then we find $v \in X_0$ uniquely determined by $\widetilde{u}$ such that

$$B_h(v, w) = \int_\Omega g(x) \cdot w(x) dx - B_h(\widetilde{u}, w)$$

Then prove two different extensions give same solution

Stability

Since $u - \widetilde{u} \in X_0$, we have

$$B_h(u, u - \widetilde{u}) = \int_\Omega g(u - \widetilde{u})dx$$

Finish after recalling $\|\cdot\|_H$ and $\|\cdot\|_X$ are equivalent in X_0

Outline

Minimization Problem

Goal: find $(\bar{u}, \bar{g}) \in (u_0 + X_0(\Omega_\delta; \mathbb{R}^n)) \times L^2(\Omega; \mathbb{R}^n)$ minimizing

$$I_\delta(u, g) = \int_\Omega F(x, u(x))dx + \frac{\lambda}{2}\|g\|^2_{L^2(\Omega;\mathbb{R}^n)}$$

subject to: $\lambda > 0$, $g \in Z_{ad} \subset L^2(\Omega; \mathbb{R}^n)$ and $u \in u_0 + X_0$ solving

$$B_h(u, v) = \int_\Omega g(x) \cdot v(x)dx \quad \forall v \in X_0$$

Minimization Problem Setup

Take Z_{ad} to be a nonempty, closed, convex, and bounded subset of $L^2(\Omega;\mathbb{R}^n)$, typically

$$Z_{ad} \; = \; \{a \leq z_i(x) \leq b, 1 \leq i \leq n\}$$

where $a \leq b$. Also, $\lambda > 0, g \in Z_{ad} \subset L^2(\Omega;\mathbb{R}^n)$, $u \in u_0 + X_0$.
Assumptions on $F : \Omega \times \mathbb{R} \to \mathbb{R}$:

1. For all $v \in \mathbb{R}$, $x \mapsto F(x, v)$ is measurable
2. For all $x \in \Omega$, $v \mapsto F(x, v)$ is continuous

Will also prove $X_0(\Omega;\mathbb{R}^n) \subset\subset L^2(\Omega;\mathbb{R}^n)$

Abstract Minimization Results

Theorem

Let Z_{ad} be a nonempty, closed, bounded, and convex subset of Z, $S : Z \to Y$ be a continuous linear operator, and $G : Y \to \mathbb{R}$ be convex and lower semi-continuous. Then the Banach Space optimization problem

$$\min_{g \in Z_{ad}} \left\{ f(g) \; := \; G(Sg) + \frac{\lambda}{2}\|g\|_Z^p \right\}$$

has an optimal solution $\bar{g}$. This solution is unique if either:

1. $\lambda > 0$
2. *S is injective and G is strictly convex*

Abstract Minimization Results (continued)

Theorem

Let Z_{ad} be a nonempty, closed, bounded, and convex subset of Z. Let $S : Z \to Y$ be a compact operator, and $G : Y \to \mathbb{R}$ be lower semi-continuous. Then the Banach Space optimization problem

$$\min_{g \in Z_{ad}} \left\{ f(g) \ := \ G(Sg) + \frac{\lambda}{2} \|g\|_Z^p \right\}$$

has an optimal solution $\bar{g}$. Furthermore, if $\lambda > 0$, and G and S are linear on their respective domains, then there is a unique minimizer

Main Compactness Result

Theorem (Compactness)

We have $X_0(\Omega; \mathbb{R}^n) \subset\subset L^2(\Omega; \mathbb{R}^n)$.

Definition (Local Compactness)

If E is a normed vector space, we call a continuous linear operator $T : E \to L^2(\mathbb{R}^n; \mathbb{R}^n)$ **locally compact** if the operator $R_K T : E \to L^2(\mathbb{R}^n; \mathbb{R}^n)$ defined via $R_K u := \mathbb{1}_K u$ is a compact operator for every compact subset $K \subset \mathbb{R}^n$.

Compactness: Strategy

- Introduce new weighted nonlocal function space, prove Hilbert
- Introduce appropriate norm and inner product
- Prove convolution lemma for matrix-weighted operators
- Prove local compactness via totally bounded approach
- Think of $j(\xi) \leftrightarrow \frac{k_\delta(\xi)}{|\xi|^2}$!

Compactness: Notation

Let $J(\xi) = \frac{\xi \otimes \xi}{|\xi|^2} j(\xi)$, where $j : \mathbb{R}^n \to [0, \infty]$ be non-negative kernel such that $j(z) = j(-z)$ for all $z \in \mathbb{R}^n$, $j \notin L^1(\mathbb{R}^n)$, and

$$0 \; < \; \int_{\mathbb{R}^n} \min\{1, |z|^2\} j(z) dz \; < \; \infty$$

Bi-linear form:

$$\mathcal{E}_j(u, v) \; := \; \frac{1}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} J(x - y)(u(x) - u(y)) \cdot (v(x) - v(y)) dxdy$$

Compactness: Notation (continued)

Energy space corresponding to $\mathcal{E}_j$:

$$\mathcal{D}^j(\mathbb{R}^n;\mathbb{R}^n) := \{u \in L^2(\mathbb{R}^n;\mathbb{R}^n), \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left|(u(x)-u(y))\cdot\frac{(x-y)}{|x-y|}\right|^2 j(x-y)dxdy < \infty\}$$

If $\Omega \subset \mathbb{R}^n$ is open:

$$\mathcal{D}^j(\Omega;\mathbb{R}^n) := \{u \in \mathcal{D}^j(\mathbb{R}^n;\mathbb{R}^n), u = 0 \text{ on } \mathbb{R}^n \setminus \Omega\}$$

Properties of $\mathcal{D}^j(\mathbb{R}^n;\mathbb{R}^n)$

Lemma

For any $\Omega \subset \mathbb{R}^n$ open, the function space $\mathcal{D}^j(\Omega;\mathbb{R}^n)$ is a Hilbert space with the associated inner product

$$\langle u, v\rangle_{\mathcal{D}^j(\Omega;\mathbb{R}^n)} := \mathcal{E}_j(u,v) + \langle u, v\rangle_{L^2(\Omega;\mathbb{R}^n)},$$

which in turn induces a norm

$$\|u\|^2_{\mathcal{D}^j(\Omega;\mathbb{R}^n)} = \mathcal{E}_j(u,u) + \|u\|^2_{L^2(\Omega;\mathbb{R}^n)}$$

Proposition

If $\mathcal{D}^j(\mathbb{R}^n;\mathbb{R}^n) \subset L^2(\mathbb{R}^n;\mathbb{R}^n)$ is a locally compact embedding, then $\mathcal{D}^j(\Omega;\mathbb{R}^n) \subset L^2(\Omega;\mathbb{R}^n)$ is compact for every bounded and open $\Omega \subset \mathbb{R}^n$.

Convolution Lemma

Lemma

Suppose $W \in L^1(\mathbb{R}^n; \mathbb{R}^{n\times n})$*. Then the corresponding convolution operator* $T_W : L^2(\mathbb{R}^n; \mathbb{R}^n) \to L^2(\mathbb{R}^n; \mathbb{R}^n)$ *defined via*

$$[T_W u]_i := \int_{\mathbb{R}^n} W_{i,\cdot}(x-y)\cdot u(y)dy = \sum_{j=1}^{n} \int_{\mathbb{R}^n} W_{i,j}(x-y)u_j(y)dy$$

for each $i \in \{1, 2, \ldots, n\}$*, is locally compact.*

Strategy: Young's Inequality for continuity, density argument, show if $M \subset L^2(\Omega; \mathbb{R}^n)$ is bounded then $T_W M$ is equi-continuous

Compactness Result Recap

Theorem

If $j : \mathbb{R}^n \to [0, \infty]$ is a non-negative kernel for which $j(z) = j(-z)$ for all $z \in \mathbb{R}^n$ and $\int_{\mathbb{R}^n} \min\{1, |z|^2\} j(z) dz < \infty$, then the embedding $\mathcal{D}^j(\mathbb{R}^n; \mathbb{R}^n) \subset L^2(\mathbb{R}^n; \mathbb{R}^n)$ is locally compact.

Pick $\tau > 0$ so $j_\tau := j \mathbb{1}_{\mathbb{R}^n \setminus B(0,\tau)} \in L^1(\mathbb{R}^n)$ and let $W := J_\tau$; define

$$J_\tau(\xi) := \frac{j_\tau}{\|j_\tau\|_{L^1(\mathbb{R}^n)}} \frac{\xi \otimes \xi}{|\xi|^2}$$

Proof of Compactness Result

We claim that

$$\int_{\mathbb{R}^n} J_\tau(\xi)d\xi \;=\; \mathbb{I}_n$$

Main diagonal:

$$\sum_{i=1}^{n} \int_{\mathbb{R}^n} \frac{j_\tau}{\|j_\tau\|_{L^1(\mathbb{R}^n)}} \frac{\xi_i^2}{|\xi|^2} d\xi \;=\; \frac{1}{\|j_\tau\|_{L^1(\mathbb{R}^n)}} \int_{\mathbb{R}^n} j_\tau(\xi)d\xi \;=\; 1$$

Off-diagonal:

$$\int_{\mathbb{R}^n} \frac{j_\tau}{\|j_\tau\|_{L^1(\mathbb{R}^n)}} \frac{\xi_i\xi_j}{|\xi|^2} d\xi \;=\; 0$$

Proof of Compactness Result (continued)

Then prove that

$$\|u - T_{J_\tau} u\|_{L^2(\mathbb{R}^n;\mathbb{R}^n)} \leq \left(\frac{2}{\|j_\tau\|_{L^1(\mathbb{R}^n)}} \right)^{\frac{1}{2}} [u]_{\mathcal{D}^j(\mathbb{R}^n;\mathbb{R}^n)}.$$

- Need $j \notin L^1(\mathbb{R}^n)$ for this bound to work
- Let $M \subset \mathcal{D}^j(\mathbb{R}^n;\mathbb{R}^n)$ bounded, $K \subset \mathbb{R}^n$ compact
- Suffices to show that $R_K(M) \subset L^2(\mathbb{R}^n;\mathbb{R}^n)$ is relatively compact (recall $R_K u = \mathbb{1}_K u$)
- The set $\widetilde{M} := [R_K T_{J_\tau}](M)$ is relatively compact in $L^2(\mathbb{R}^n;\mathbb{R}^n)$
- Use convolution lemma and a totally bounded approach

Existence of Minimizers

Theorem (Existence of Minimizers)

There exists $\bar{g} \in L^2(\Omega; \mathbb{R}^n)$ *minimizing*

$$I_\delta(u, g) = \int_\Omega F(x, u(x))dx + \frac{\lambda}{2}\|g\|^2_{L^2(\Omega;\mathbb{R}^n)},$$

where $u \in u_0 + X_0$ *solves*

$$B_h(u, v) = \int_\Omega g(x) \cdot v(x)dx \quad \forall v \in X_0$$

Can use compactness to invoke abstract minimization results

Uniqueness of Minimizers

Theorem (Uniqueness of Minimizers)

The minimizer that exists is unique if F is linear in its second argument:

$$F(x, \alpha u(x) + \beta v(x)) \;=\; \alpha F(x, u(x)) + \beta F(x, v(x))$$

for $\alpha, \beta \in \mathbb{R}$

Prove indirectly, using strict convexity of $g \to \|g\|^2_{L^2(\Omega;\mathbb{R}^n)}$

Outline

Local and Nonlocal Energies

Define these for $u \in L^2(\Omega; \mathbb{R}^n)$:

$$E_\delta(u) := \int_{\Omega_\delta} \int_{\Omega_\delta} H(x,y) k_\delta(x-y) \frac{|Du(x,y)|^2}{|x-y|^2} dxdy;$$

$$E_0(u) := \frac{1}{(n+2)(n+4)} \int_\Omega h(x)(2\|\mathrm{Sym}(\bigtriangledown u(x))\|_F^2 + \mathrm{div}(u(x))^2) dx$$

Take to be $+\infty$ when not well-defined

Minimization of Local Functional

Theorem

Suppose $\{(\bar{u}_\delta, \bar{g}_\delta)\}_{\delta>0}$ denotes the sequence of minimizers for the functionals $\{I_\delta\}_{\delta>0}$. If $\bar{u}_\delta \to \bar{u}$ strongly in $L^2(\Omega;\mathbb{R}^n)$ and $\bar{g}_\delta \rightharpoonup \bar{g}$ weakly in $L^2(\Omega;\mathbb{R}^n)$, then $(\bar{u}, \bar{g})$ is a minimizer to the local optimal control problem.

Notice $\{\bar{u}_\delta\}_{\delta>0}$ have bounded semi-norm so compactness gives a $\bar{u}$
Notice $\{\bar{g}_\delta\}_{\delta>0}$ are bounded in $L^2(\Omega;\mathbb{R}^n)$ so compactness gives a $\bar{g}$
Need minimizers to be preserved in limit!

Γ-Convergence

Definition

We say that the family $E_\delta : L^2(\Omega;\mathbb{R}^n) \to \mathbb{R} \cup \{+\infty\}$ **Γ-converges** strongly in $L^2(\Omega;\mathbb{R}^n)$ to $E_0 : L^2(\Omega;\mathbb{R}^n) \to \mathbb{R} \cup \{+\infty\}$ (denoted $E_\delta \xrightarrow{\Gamma} E_0$) if:

i) **The liminf inequality:** Assume $u_\delta \to u$ strongly in $L^2(\Omega;\mathbb{R}^n)$. Then

$$E_0(u) \leq \liminf_{\delta\to 0^+} E_\delta(u_\delta)$$

ii) **Recovery sequence property:** For each $u \in L^2(\Omega;\mathbb{R}^n)$, there exists a sequence $\{u_\delta\}_{\delta>0}$ where $u_\delta \to u$ strongly in $L^2(\Omega;\mathbb{R}^n)$ and

$$\limsup_{\delta\to 0^+} E_\delta(u_\delta) \leq E_0(u)$$

Quantitative L^2-continuity

Lemma (Standard L^2-continuity)

If $v \in L^2(\Omega;\mathbb{R}^n)$, then

$$\lim_{h\to 0}\int_{\mathbb{R}^n} |v(x+h)-v(x)|^2 dx \;=\; 0.$$

Lemma (Quantitative L^2-continuity)

Fix $\xi \in \mathbb{R}^n \setminus \{0\}$, and let $v \in W_0^{1,2}(\mathbb{R}^n;\mathbb{R}^n)$. Then we have

$$\int_{\mathbb{R}^n} \left|(v(y+\xi)-v(y))\cdot\frac{\xi}{|\xi|}\right|^2 dy \le |\xi|^2 \|Sym(\nabla v)\|^2_{L^2(\mathbb{R}^n;\mathbb{R}^{n\times n})}$$

Other Technical Lemmas

Lemma (Continuous Embedding)

There exists a $\delta_0 > 0$ *such that for all* $\delta \in (0, \delta_0)$,

$$\|v\|_{X(\Omega_\delta;\mathbb{R}^n)} \leq \|v\|_{W^{1,2}(\Omega;\mathbb{R}^n)}.$$

That is, $W_0^{1,2}(\Omega;\mathbb{R}^n) \hookrightarrow X_0(\Omega_\delta;\mathbb{R}^n)$ *for all* $\delta > 0$ *sufficiently small.*

Lemma (Difference of Squares)

For any $\delta > 0$*, we have these inequalities for all* $u, v \in X(\Omega_\delta;\mathbb{R}^n)$*:*

$$E_\delta(u) - E_\delta(v) \leq E_\delta(u+v)E_\delta(u-v);$$

$$\left| \|u\|^2_{X(\Omega_\delta;\mathbb{R}^n)} - \|v\|^2_{X(\Omega_\delta;\mathbb{R}^n)} \right| \leq \|u+v\|_{X(\Omega_\delta;\mathbb{R}^n)} \|u-v\|_{X(\Omega_\delta;\mathbb{R}^n)}$$

Useful Integral Lemma

Lemma

For any $A \in \mathbb{R}^{n \times n}$,

$$\frac{1}{\alpha(n)} \int_{\mathbb{S}^{n-1}} |\langle A\omega, \omega\rangle|^2 d\sigma(\omega) = \frac{1}{(n+2)(n+4)}(2\|A\|_F^2 + \mathrm{Tr}(A)^2).$$

- Expand the inner product and square it
- Casework based on which indices of quadruple sum are equal
- Calculate each sum

Highlights: Recovery Sequence Proof

Use Taylor and symmetry of H, focus on controlling

$$C \int_{\Omega_\delta} \int_{\Omega_\delta} h(y) k_\delta(x-y)|x-y| dxdy;$$

$$\int_{\Omega_\delta} \int_{\Omega_\delta} h(y) k_\delta(x-y) \left\langle \mathrm{Sym}(\triangledown u(x)) \frac{x-y}{|x-y|}, \frac{x-y}{|x-y|} \right\rangle^2 dxdy$$

Split first integral into cases: $|x-y| > a$, $|x-y| < a$, for $a > 0$ arbitrary

Handle second integral with Fubini, change of variables, integral lemma with $A := \mathrm{Sym}(\triangledown u(x))$

Recovery sequence is $u_\delta := u$!

Needed Compactness Result

Proposition (Mengesha-Du 2014 ($p = 2$))

Suppose $\{u_\delta\}_{\delta>0} \subset L^2(\Omega;\mathbb{R}^n)$ *is a bounded family of vector fields such that*

$$\sup_{\delta>0} \int_\Omega \int_\Omega k_\delta(x-y) \frac{|\mathcal{D}u_\delta(x,y)|^2}{|x-y|^2} dxdy \; < \; \infty,$$

where $k_\delta(r)r^{-2}$ *is nonincreasing; then the family has compact closure in* $L^2(\Omega;\mathbb{R}^n)$*, and any limit point* u *belongs to* $W^{1,2}(\Omega;\mathbb{R}^n)$

Highlights: Lim-inf Inequality Proof

$$E_0(u) \leq \liminf_{\delta \to 0^+} E_\delta(u_\delta)$$

- Assume without loss of generality $\liminf_{\delta \to 0^+} E_\delta(u_\delta) < \infty$
- Use compactness, conclude limit point $u \in W^{1,2}(\Omega; \mathbb{R}^n)$
- Prove liminf inequality for the following forms of h: indicator functions; simple functions; non-negative $L^\infty(\Omega)$ functions

Highlights: Lim-inf Inequality Proof (continued)

Liminf on $A \times \Omega$ for $h(x) = \chi_A(x)$:

$$\frac{1}{(n+2)(n+4)} \int_A 2\|\text{Sym}(\triangledown u(x))\|_F^2 + \text{div}(u(x))^2 dx \;\leq$$
$$\text{liminf}_{\delta \to 0^+} \int_A \int_{\Omega_\delta} k_\delta(x-y) \frac{|Du_\delta(x,y)|^2}{|x-y|^2} dxdy$$

Convolution approach: let $w_{\epsilon,\delta} := \eta_\epsilon * u_\delta$ and $w_\epsilon := \eta_\epsilon * u$, show $w_{\epsilon,\delta} \to w_\epsilon$ in $C^1(\bar{A};\mathbb{R}^n)$ as $\delta \to 0^+$.

Highlights: Lim-inf Inequality Proof (continued)

Intermediate step (using $h(x) = \chi_A(x)$):

$$\int_A \int_\Omega k_\delta(x-y) \frac{|Dw_{\epsilon,\delta}(x,y)|^2}{|x-y|^2} dxdy \le \int_A \int_\Omega k_\delta(x-y) \frac{|Du_\delta(x,y)|^2}{|x-y|^2} dxdy$$

Send $\delta \to 0^+$ for fixed $\epsilon > 0$:

$$\frac{1}{(n+2)(n+4)} \int_A (2\|\text{Sym}(\nabla w_\epsilon)\|_F^2 + \text{div}(w_\epsilon(x))^2 dx \le$$
$$\lim_{\delta \to 0^+} \int_A \int_\Omega k_\delta(x-y) \frac{|Dw_{\epsilon,\delta}(x,y)|^2}{|x-y|^2} dxdy$$

Finally send $\epsilon \to 0^+$, use Fatou

Preservation of Minimizers

Lemma

If $\{v_\delta\}_{\delta>0}$ is a sequence of minimizers for $\{E_\delta\}_{\delta>0}$ over $L^2(\Omega;\mathbb{R}^n)$, v is a limit point of this sequence, and $E_\delta \xrightarrow{\Gamma} E_0$, then v is a minimizer of E_0 on $L^2(\Omega;\mathbb{R}^n)$. Finally,

$$\lim_{\delta\to 0^+} E_\delta(v_\delta) = E_0(v)$$

Minimization with Source Term

Define these functionals on $L^2(\Omega; \mathbb{R}^n)$:

$$\widetilde{E_\delta}(u) := E_\delta(u) + \int_\Omega g(x) \cdot u(x) dx;$$

$$\widetilde{E_0}(u) := E_0(u) + \int_\Omega g(x) \cdot u(x) dx$$

considering them to be $+\infty$ when not well-defined; then $\widetilde{E_\delta} \xrightarrow{\Gamma} \widetilde{E_0}$, and minimizers are still preserved

Tying Together Optimal Control

Since $(\bar{u}_\delta, \bar{g}_\delta) \in \mathcal{A}^\delta$ for each $\delta > 0$, we have $\widetilde{E_\delta}(\bar{u}_\delta) \leq 0$; rearrange and use nonlocal Poincaré to get

$$[\bar{u}_\delta]_{X(\Omega_\delta;\mathbb{R}^n)} \;\leq\; C\|\bar{g}_\delta\|_{L^2(\Omega;\mathbb{R}^n)} \;\leq\; M$$

Finally, for a given $(f, v) \in \mathcal{A}^{\text{loc}}$,

$$I_0(\bar{u}, \bar{g}) \;\leq\; \lim_{\delta\to 0^+} I_\delta(\bar{u}_\delta, \bar{g}_\delta) \;\leq\; \lim_{\delta\to 0^+} I_\delta(f, v_\delta) \;\leq\; I_0(f, v)$$

Outline

Possible Next Steps

Numerics: Simulations/computing, finite element approximations

Variants on the theory: Consider other linear PD problems, nonlinear problems

Optimal Design: Perform optimization over different choices of design function H

What happens when $p = 1$?

Theorem (Ponce 2003)

Let $n \geq 2$, and suppose $\{f_\delta\}_{\delta>0} \subset L^p(\Omega)$ is a bounded sequence for which

$$\sup_{\delta>0} \int_\Omega \int_\Omega k_\delta(x-y) \frac{|f_\delta(x) - f_\delta(y)|^p}{|x-y|^p} dxdy < \infty,$$

then $\{f_\delta\}_{\delta>0}$ is relatively compact in $L^p(\Omega)$.
If $f_{\delta_m} \to f$ strongly in $L^p(\Omega)$, then

- *If $p \in (1, \infty)$, then $f \in W^{1,p}(\Omega)$*
- *If $p = 1$, then $f \in BV(\Omega)$.*

Conjectures I/II: Density

Conjecture (Global Approximation in $X_0(\Omega;\mathbb{R}^n)$)

Let $\Omega \subset \mathbb{R}^n$ be an open set with continuous boundary. Then for any $u \in X_0(\Omega;\mathbb{R}^n)$, there exists a family $\{\rho_\epsilon\}_{\epsilon>0}$ such that $\|\rho_\epsilon - u\|_{X(\Omega;\mathbb{R}^n)} \to 0$ as $\epsilon \to 0^+$. That is, $C_0^\infty(\Omega;\mathbb{R}^n)$ is a dense subset of $X_0(\Omega;\mathbb{R}^n)$.

Conjecture (Global Approximation in $X(\Omega;\mathbb{R}^n)$)

Suppose that $|\xi|^{-2}k_\delta(\xi) \in L^1_{loc}(\mathbb{R}^n)$, and let $\Omega \subset \mathbb{R}^n$ be an open set with continuous boundary. Then for any $u \in X(\Omega;\mathbb{R}^n)$, there exists a family $\{\rho_\epsilon\}_{\epsilon>0}$ such that $\|\rho_\epsilon - u\|_{X(\Omega;\mathbb{R}^n)} \to 0$ as $\epsilon \to 0^+$. That is, $C^\infty(\Omega;\mathbb{R}^n)$ is a dense subset of $X(\Omega;\mathbb{R}^n)$.

Conjecture III: Extension on X

Conjecture (Extension Lemma on X)

Let $u \in X(\Omega_\delta;\mathbb{R}^n)$, where $\partial\Omega$ is a C^1 boundary. Suppose V is a bounded, open set such that $\Omega_\delta \subset\subset V$. Then there exists a bounded linear [extension] operator $E : X(\Omega_\delta;\mathbb{R}^n) \to X(\mathbb{R}^n;\mathbb{R}^n)$ such that $Eu = u$ a.e. in Ω_δ, Eu is supported in V, and

$$\|Eu\|_{X(\mathbb{R}^n;\mathbb{R}^n)} \le C\|u\|_{X(\Omega_\delta;\mathbb{R}^n)}.$$

Here the constant $C > 0$ depends only on Ω_δ, and V.

- Cannot extend by zero anymore
- Anticipate may require density result to be true

Outline

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