EXPLICIT FORMULAS CATALOGUE: CONVERTING BETWEEN BALLS AND BOUNDARIES

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1. Introduction

Many times in PDE (and analysis) for that matter we wish to prove or utilize identities where one side of the identity involves the integral over an open ball and the other involves the integral over the boundary of a ball. These sometimes have nice geometric interpretations but the focus is usually in using them to solve analytic problems related to energy methods or a priori estimates for solutions to certain PDEs. We present the theorems roughly as they appear in [Ev]. Throughout, Ω denotes an open bounded set in $\mathbb{R}^n$ and ν denotes a unit outward normal.

2. Some Theorems

Theorem 2.1 (Gauss-Green)**.** *(Evans p. 711) Let $u \in C^1(\bar{\Omega})$. Then*

$$\int_\Omega u_{x_i} dx = \int_{\partial\Omega} u\nu^i dS$$

for all $1 \leq i \leq n$.

Theorem 2.2 (Divergence Theorem)**.** *(Evans p. 712) Let u be an n-dimensional vector field where each component is $C^1(\bar{\Omega})$. Then*

$$\int_\Omega divu dx = \int_{\partial\Omega} u \cdot \nu dS$$

Theorem 2.3 (Integration by Parts)**.** *(Evans p. 712) Let $u, v \in C^1(\bar{\Omega})$. Then*

$$\int_\Omega u_{x_i} v dx = -\int_\Omega u v_{x_i} dx + \int_{\partial\Omega} uv\nu^i dS$$

for all $1 \leq i \leq n$.

Theorem 2.4 (Green Formulas)**.** *(Evans p. 712) Let $u, v \in C^2(\bar{\Omega})$. Then*

$$\int_\Omega \triangle u dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} dS$$

$$\int_\Omega \bigtriangledown v \cdot \bigtriangledown u dx = -\int_\Omega u \triangle v dx + \int_\Omega \frac{\partial v}{\partial \nu} u dS$$

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$$\int_{\Omega} u \bigtriangledown v - v \bigtriangledown u dx = \int_{\partial\Omega} u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} dS$$

Theorem 2.5. *(Evans p. 712) Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous and in $L^1(\mathbb{R})$. Then*

$$\int_{\mathbb{R}^n} f dx = \int_0^\infty \left(\int_{\partial B(x_0,r)} f dS \right) dr$$

for each $x_0 \in \mathbb{R}^n$. Furthermore, for each $r > 0$,

$$\frac{d}{dr} \left(\int_{B(x_0,r)} f dx \right) = \int_{\partial B(x_0,r)} f dS$$

Theorem 2.6 (Moving Regions)**.** *(Evans p. 713) Let $\Omega(\tau) \subset \mathbb{R}^n$ be a smoothly parameterized family of moving regions that are bounded. Let v be the velocity of the moving boundary $\partial\Omega(\tau)$. Then for a smooth function f in x and τ,*

$$\frac{d}{d\tau} \int_{\Omega(\tau)} f dx = \int_{\partial\Omega(\tau)} f v \cdot \nu dS + \int_{\Omega(\tau)} f_\tau dx$$

Theorem 2.7 (Reverse Differentiation)**.** *(Han p. 231) If $\psi : \mathbb{R}^n \to \mathbb{R}$ is a smooth, compactly supported function, then for $t > 0$ and $x, \omega \in \mathbb{R}^n$,*

$$\psi(x + t\omega) = - \int_t^\infty \frac{\partial}{\partial s} \psi(x + s\omega) ds$$

References

[Ev] L.C. Evans, *Partial Differential Equations*, 2nd Edition, American Mathematical Society, 2013.

[Han] Q. Han, *A Basic Course in Partial Differential Equations*, 1st Edition, Americal Mathematical Society, 2011.