EXPLICIT FORMULAS CATALOGUE: SECOND-ORDER LINEAR PDE

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1. Introduction

This document is designed as a fast reference for theorems and lemmas pertaining to explicit solutions of certain second-order PDE. The ordering closely follows [Ev] but this guide is also a suitable companion to [Han]. The focus is on the Laplace, heat, and wave equations, including different cases as the number of dimensions changes. There is also a brief discussion of Duhamel's Principle to convert knowledge of solutions to some homogeneous problems to their inhomogeneous variants.

Throughout $\alpha(n)$ denotes the volume of the unit ball in $\mathbb{R}^n$, while f and g represent continuous functions on the respective spatial or spatial-temporal domains. Assume u has any needed regularity, which is $u \in C^2(\mathbb{R}^n)$ for spatial problems and $u \in C^2(\mathbb{R}^n \times \{t > 0\})$ for spatial-temporal problems. As a final remark, the inhomogeneous problem solutions can be derived from the solutions to the homogeneous problems for the same PDE using Duhamel's Principle.

2. Laplace/Poisson Equation

The following are solutions to $\triangle u = 0$ in $\mathbb{R}^n \setminus \{0\}$:
Fundamental Solution for $n = 2$

$$u(x) = -\frac{1}{2\pi} \ln|x|$$

Fundamental Solution for $n \geq 3$

$$u(x) = \frac{1}{n(n-2)\alpha(n)|x|^{n-2}}$$

The following are solutions to $-\triangle u = f$ in $\mathbb{R}^n$:
Poisson Equation Solution, $n = 2$

$$u(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \ln(|x-y|) f(y) dy$$

Poisson Equation Solution, $n \geq 3$

$$u(x) = \frac{1}{n(n-2)\alpha(n)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}} dy$$

Poisson Solution

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Poisson Equation on a Ball (let $R > 0$): the solution to the PDE

$$\begin{cases} \triangle u = 0, x \in B(0,R) \\ u = g, x \in \partial B(0,R) \end{cases}$$

is

$$u(x) = \frac{R^2 - |x|^2}{n\alpha(n)R} \int_{\partial B(0,R)} \frac{g(y)}{|y-x|^n} dS(y)$$

and the *Poisson Kernel* is

$$K(x,y) = \frac{R^2 - |x|^2}{n\alpha(n)R} \cdot \frac{1}{|x-y|^n}$$

3. Homogeneous Heat Equation

General PDE for homogeneous variant:

$$\begin{cases} u_t - \triangle u = 0, (x,t) \in \mathbb{R}^n \times \{t > 0\} \\ u = f(x), \qquad (x,t) \in \mathbb{R}^n \times \{t = 0\} \end{cases}$$

Fundamental Solution ($f = 0$)

$$u(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4t}}$$

NOTE: for this PDE we conventionally define the Fundamental Solution to be 0 for $t < 0$; this one solution does not take into account any initial conditions and so its direct use is more limited in problem-solving.

Homogeneous Problem

$$u(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} f(y) dy$$

4. Inhomogeneous Heat Equation

General PDE for inhomogeneous variant:

$$\begin{cases} u_t - \triangle u = f(x,t), (x,t) \in \mathbb{R}^n \times \{t > 0\} \\ u = 0, \qquad (x,t) \in \mathbb{R}^n \times \{t = 0\} \end{cases}$$

Inhomogeneous Problem

$$u(x,t) = \int_0^t \frac{1}{(4\pi(t-s))^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4(t-s)}} f(y,s) dy ds$$

5. Homogeneous Wave Equation

General PDE for homogeneous variant:

$$\begin{cases} u_t - \triangle u = 0, (x,t) \in \mathbb{R}^n \times \{t > 0\} \\ u = f(x), \qquad (x,t) \in \mathbb{R}^n \times \{t = 0\} \\ u_t = g(x), \quad (x,t) \in \mathbb{R}^n \times \{t = 0\} \end{cases}$$

Homogeneous Problem for $n = 1$ (D'Alembert Formula)

$$u(x,t) = \frac{1}{2}(f(x+t) + f(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} g(s)ds$$

Homogeneous Problem for $n = 2$ (Poisson Formula)

$$u(x,t) = \frac{1}{2} \fint_{B(x,t)} \frac{tf(y) + t^2 g(y) + t \bigtriangledown f(y) \cdot (y-x)}{(t^2 - |y-x|^2)^{\frac{1}{2}}} dy$$

Homogeneous Problem for $n = 3$ (Kirchhoff Formula)

$$u(x,t) = \fint_{\partial B(x,t)} tg(y) + f(y) + \bigtriangledown f(y) \cdot (y-x) dS(y)$$

6. Inhomogeneous Wave Equation

General PDE for inhomogeneous variant:

$$\begin{cases} u_{tt} - \triangle u = f(x,t), (x,t) \in \mathbb{R}^n \times \{t > 0\} \\ u = 0, \qquad\qquad (x,t) \in \mathbb{R}^n \times \{t = 0\} \\ u_t = 0, \qquad\qquad (x,t) \in \mathbb{R}^n \times \{t = 0\} \end{cases}$$

Inhomogeneous Problem for $n = 1$

$$u(x,t) = \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} f(y,s)dyds = \frac{1}{2}\int_0^t \int_{x-s}^{x+s} f(y,t-s)dyds$$

Inhomogeneous Problem for $n = 2$

$$u(x,t) = \frac{1}{2\pi}\int_0^t \int_{B(x,t-s)} \frac{f(y,s)}{((t-s)^2 - |y-x|^2)^{\frac{1}{2}}} dyds$$

Inhomogeneous Problem for $n = 3$

$$u(x,t) = \int_0^t \fint_{\partial B(x,t-s)} (t-s) f(y,s) dS(y) ds = \frac{1}{4\pi}\int_{B(x,t)} \frac{f(y, t - |y-x|)}{|y-x|} dy$$

References

[Ev] L.C. Evans, *Partial Differential Equations*, 2nd Edition, American Mathematical Society, 2013.
[Han] Q. Han, *A Basic Course in Partial Differential Equations*, 1st Edition, Americal Mathematical Society, 2011.