MAXIMA PRINCIPLE CATALOGUE: ELLIPTIC SPATIAL PROBLEMS

JOSHUA M. SIKTAR

1. Introduction

This is a catalogue of different maximum principle theorems, both strong and weak. This document lists out those principles stated and proven in [Ev] and [Han], and is designed as a concise reference of these results for those looking for a suitable maximum principle to solve a given problem, be it an a priori estimate, an energy method, or some other application. The emphasis on this document is on results for elliptic operators in purely spatial dimensions; in particular, results pertaining to the wave and heat equations are omitted. Finally, all maximum principles listed have corresponding minimum principles by replacing the functions in question with their opposites.

2. Maximum Principles with Laplacian

Theorem 2.1 (Evans p. 27)**.** *Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ be harmonic in Ω. Then:*

(1) $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$.
(2) *If Ω is connected and there exists a point $x_0 \in \Omega$ such that $u(x_0) = \max_{\bar{\Omega}} u$, then u is constant in Ω.*

3. Maximum Principles for $\triangle u + cu$

Theorem 3.1 (Han p. 113)**.** *Let Ω be a bounded domain in $\mathbb{R}^n$ and c be a continuous function in Ω with $c \leq 0$. Suppose $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is such that $\triangle u + cu \geq 0$ in Ω. Then u attains on $\partial\Omega$ its nonnegative maximum in $\bar{\Omega}$, i.e. $\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u^+$.*

Corollary 3.2 (Han p. 114)**.** *Let Ω be a bounded domain in $\mathbb{R}^n$ and c be a continuous function in Ω with $c \leq 0$. Suppose $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is such that $\triangle u + cu \geq 0$ in Ω. If we further assume $u \leq 0$ on $\partial\Omega$, then $u \leq 0$ in Ω.*

Corollary 3.3 (Han p. 114)**.** *Let Ω be a bounded domain in $\mathbb{R}^n$ and c be a continuous function in Ω with $c \leq 0$. Suppose $u, v \in C^2(\Omega) \cap C(\bar{\Omega})$ is such that $\triangle u + cu \geq \triangle v + cv$ in Ω. If we further assume $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.*

Theorem 3.4 (Han p. 117)**.** *Let Ω be a bounded domain in $\mathbb{R}^n$ and c be a continuous function in Ω with $c \leq 0$. Suppose $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is such that $\triangle u + cu \geq 0$ in Ω. Then u attains only on $\partial\Omega$ its nonnegative maximum in $\bar{\Omega}$ unless u is a constant.*

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Corollary 3.5 (Han p. 114)**.** *Let Ω be a bounded domain in $\mathbb{R}^n$ and c be a continuous function in Ω with $c \leq 0$. Suppose $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is such that $\triangle u + cu \geq 0$ in Ω. If we further assume $u \leq 0$ on $\partial\Omega$, then $u < 0$ in Ω or u is a nonpositive constant in Ω.*

4. Maximum Principles for General Elliptic Operators

We consider operators of the *nondivergence form*

$$Lu = -\sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}u_{x_i x_j} + \sum_{i=1}^{n} b_i u_{x_i} + cu$$

in this section of the document.

These theorems are referred to as Weak Maximum Principles.

Theorem 4.1 (Evans p. 344)**.** *Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ and $c = 0$ on Ω. Then:*

(1) *If $Lu \leq 0$ in Ω, then $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$.*
(2) *If $Lu \geq 0$ in Ω, then $\min_{\bar{\Omega}} u = \min_{\partial\Omega} u$.*

Theorem 4.2 (Evans p. 346)**.** *Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ and $c \geq 0$ on Ω. Then:*

(1) *If $Lu \leq 0$ in Ω, then $\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u^+$.*
(2) *If $Lu \geq 0$ in Ω, then $\min_{\bar{\Omega}} u \geq \min_{\partial\Omega} u^-$.*

In particular if $Lu = 0$ in Ω then $\max_{\Omega}|u| = \max_{\partial\Omega}|u|$.

This is called the *Hopf Lemma*.

Lemma 4.3 (Evans p. 347)**.** *Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ and $c = 0$ on Ω. Suppose further $Lu \leq 0$ in Ω and $\exists x_0 \in \partial\Omega$ where $u(x_0) > u(x)$ for all $x \in \Omega$. Finally assume U satisfies the interior ball condition at x_0 ($\partial\Omega$ being a C^2 boundary is sufficient). Then:*

(1) $\frac{\partial u}{\partial n}(x_0) > 0$.
(2) *We can also obtain the above result if $c \geq 0$ in Ω and $u(x_0) \geq 0$.*

These theorems are referred to as Strong Maximum Principles.

Theorem 4.4 (Evans p. 349)**.** *Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ and $c = 0$ on Ω, where Ω is open, connected, and bounded. Then:*

(1) *If $Lu \leq 0$ in Ω and u attains a maximum over $\bar{\Omega}$ at an interior point then u is constant in Ω.*
(2) *If $Lu \geq 0$ in Ω and u attains a minimum over $\bar{\Omega}$ at an interior point then u is constant in Ω.*

Theorem 4.5 (Evans p. 350)**.** *Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ and $c \geq 0$ on Ω, where Ω is open, connected, and bounded. Then:*

(1) *If $Lu \leq 0$ in Ω and u attains a nonnegative maximum over $\bar{\Omega}$ at an interior point then u is constant in Ω.*
(2) *If $Lu \geq 0$ in Ω and u attains a nonpositive minimum over $\bar{\Omega}$ at an interior point then u is constant in Ω.*

5. Harnack Inequalities

We consider operators of the *nondivergence form*

$$Lu = -\sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j} u_{x_i x_j} + \sum_{i=1}^{n} b_i u_{x_i} + cu$$

in this section of the document.

Theorem 5.1 (Evans p. 351)**.** *Assume $u \geq 0$ is a C^2 solution of $Lu = 0$ in Ω, and let $V \subset\subset U$ be connected. Then $\exists C > 0$ depending on V and n such that $\sup_V u \leq C \inf_V u$.*

Note: The above is stated and proven for u harmonic on Evans p. 32, and Han p. 110.

Theorem 5.2 (Evans p. 86)**.** *Let u be positive and harmonic in $B(0,r)$. Then*

$$r^{n-2}\frac{r-|x|}{(r+|x|)^{n-1}}u(0) \leq u(x) \leq r^{n-2}\frac{r+|x|}{(r-|x|)^{n-1}}u(0).$$

Note: This is listed as an exercise in Evans.

References

[Ev] L.C. Evans, *Partial Differential Equations*, 2nd Edition, American Mathematical Society, 2013.
[Han] Q. Han, *A Basic Course in Partial Differential Equations*, 1st Edition, Americal Mathematical Society, 2011.