MAXIMA PRINCIPLE CATALOGUE: PARABOLIC PROBLEMS

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1. Introduction

This is a catalogue of different maximum principle theorems, both strong and weak. This document lists out those principles stated and proven in [Ev] and [Han], and is designed as a concise reference of these results for those looking for a suitable maximum principle to solve a given problem, be it an a priori estimate, an energy method, or some other application. The emphasis on this document is on results for parabolic PDEs, specifically the heat equation. Finally, all maximum principles listed have corresponding minimum principles by replacing the functions in question with their opposites.

2. Maximum Principles for $u_t - \triangle u$

Theorem 2.1 (Evans p. 55)**.** *Assume $u \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega_T})$ solves the heat equation in Ω_T. Then $\max_{\overline{\Omega_T}} u = \max_{\Gamma_T} u$. Furthermore, if U is connected and there exists a point $(x_0, t_0) \in \Omega_T$ such that $u(x_0, t_0) = \max_{\overline{\Omega_T}} u$, then u is constant in $\Omega \times [0, t_0]$.*

Theorem 2.2 (Evans p. 57)**.** *Assume $u \in C^{2,1}(\mathbb{R}^n \times (0, T]) \cap C(\mathbb{R}^n \times [0, T])$ solves the heat equation in $\mathbb{R}^n \times (0, T)$ with continuous initial data $u = g$ on $\mathbb{R}^n \times \{t = 0\}$. If u satisfies the growth estimate $u(x, t) \leq Ae^{a|x|^2}$ on $\mathbb{R}^n \times [0, T]$ for some $a, A > 0$, then $\sup_{\mathbb{R}^n \times [0,T]} u = \sup_{\mathbb{R}^n} g$.*

3. Maximum Principles for General Parabolic Operators

We consider operators of the *nondivergence form*

$$Lu = -\sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j} u_{x_i x_j} + \sum_{i=1}^{n} b_i u_{x_i} + cu$$

in this section of the document.

These theorems are referred to as Weak Maximum Principles.

Theorem 3.1 (Evans p. 389)**.** *Assume $u \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega_T})$ and L is a parabolic operator where c vanishes on Ω_T. Then*

(1) *If $u_t + Lu \leq 0$ on Ω_T then $\max_{\bar{\Omega_T}} u = \max_{\Gamma_T} u$.*
(2) *If $u_t + Lu \geq 0$ on Ω_T then $\min_{\bar{\Omega_T}} u = \min_{\Gamma_T} u$.*

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Theorem 3.2 (Evans p. 390)**.** *Assume $u \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega_T})$ and L is a parabolic operator where c is continuous on Ω_T. Then*

(1) *If $u_t - Lu \leq 0$ on Ω_T and $u \leq 0$ on Γ_T then $u \leq 0$ on $\bar{\Omega_T}$.*
(2) *If $u_t - Lu \geq 0$ on Ω_T and $u \geq 0$ on Γ_T then $u \geq 0$ on $\bar{\Omega_T}$.*
(3) *If $u_t - Lu = 0$ on Ω_T and $u = 0$ on Γ_T then $u = 0$ on $\bar{\Omega_T}$.*

Theorem 3.3. *Assume $u \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega_T})$ and L is a parabolic operator where $c \geq 0$ on Ω_T. Then*

(1) *If $u_t + Lu \leq 0$ on Ω_T then $\max_{\bar{\Omega_T}} u \leq \max_{\Gamma_T} u^+$.*
(2) *If $u_t + Lu \geq 0$ on Ω_T then $\min_{\bar{\Omega_T}} u \geq \min_{\Gamma_T} u^-$.*

This is called a Strong Maximum Principle.

Theorem 3.4 (Evans p. 397)**.** *Assume $u \in C^{2,1}(\Omega_T) \cap C(\overline{\Omega_T})$ and L is a parabolic operator where $c \geq 0$ on Ω_T. Also assume U is connected.*

(1) *If $u_t + Lu \leq 0$ in Ω_T and u attains a nonnegative maximum over $\bar{\Omega_T}$ at an interior point $(x_0, t_0) \in \Omega_T$, then u is constant in $\Omega \times (0, t_0)$.*
(2) *If $u_t + Lu \geq 0$ in Ω_T and u attains a nonpositive maximum over $\bar{\Omega_T}$ at an interior point $(x_0, t_0) \in \Omega_T$, then u is constant in $\Omega \times (0, t_0)$.*

Note: Han's book [Han] also covers some of these principles specifically for operators of the form $\triangle u - cu$.

4. Harnack Inequalities

This is called the **Parabolic Harnack Inequality**.

Theorem 4.1 (Evans p. 391)**.** *Assume $u \in C^{2,1}(\Omega_T)$ solves $u_t + Lu = 0$ in Ω_T and $u \geq 0$ in Ω_T. Suppose V is a connected, compactly contained set in U. Then for each $0 < t_1 < t_2 \leq T$, there exists a constant C such that*

$$\sup_V u(\cdot, t_1) \leq C \inf_V u(\cdot, t_2)$$

where C depends only on V, t_1, t_2, and the coefficients of L.

References

[Ev] L.C. Evans, *Partial Differential Equations*, 2nd Edition, American Mathematical Society, 2013.
[Han] Q. Han, *A Basic Course in Partial Differential Equations*, 1st Edition, Americal Mathematical Society, 2011.