PROPERTIES OF HARMONIC FUNCTIONS CATALOGUE

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1. Introduction

This document is designed as a fast reference for theorems and lemmas pertaining to properties of harmonic functions in $\mathbb{R}^n$. The ordering closely follows [Ev] but this guide is also a suitable companion to [Han]. The results are listed in thematic categories, but they all have to do with harmonic functions in only spatial dimensions (as well as sub/superharmonic functions). Throughout $\alpha(n)$ denotes the volume of the unit ball in $\mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$ is an open, bounded set.

2. Terminology

Definition 2.1. *A function u is harmonic if $\triangle u = 0$, subharmonic if $\triangle u \geq 0$, and superharmonic if $\triangle u \leq 0$.*

3. Solutions

Lemma 3.1 (One-Dimensional)**.** *The only solutions to Laplace's Equation in $\mathbb{R}^1$ are constant and linear functions.*

Lemma 3.2 (Fundamental Solution)**.** *(Evans p. 22) The fundamental solution to Laplace's Equation $\triangle u = 0$ is $-\frac{1}{2\pi}\ln|x|$ on $\mathbb{R}^2 \setminus \{0\}$ and $\frac{1}{n(n-2)\alpha(n)}\frac{1}{|x|^{n-2}}$ on $\mathbb{R}^n \setminus \{0\}$ for any $n \geq 3$.*

Theorem 3.3 (Solution to Poisson's Equation)**.** *(Evans p. 23) The solution to Poisson's Equation $-\triangle u = f$ in $\mathbb{R}^n$ is*

$$-\frac{1}{2\pi}\int_{\mathbb{R}^2} \ln(|x-y|)f(y)dy$$

if $n = 2$ and

$$\frac{1}{n(n-2)\alpha(n)}\int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}}dy$$

if $n \geq 3$.

Lemma 3.4. *(Evans p. 85) If $u \in C^2(\bar{\Omega})$ is harmonic then the function $v := |Du|^2$ is subharmonic.*

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4. Mean-Value Properties

Theorem 4.1 (Mean-Value Property)**.** *(Evans p. 25) If $u \in C^2(\Omega)$ is harmonic then for any open ball $B(x,r) \in \Omega$,*

$$u(x) = \fint_{\partial B(x,r)} u(y)dS = \fint_{B(x,r)} u(y)dy$$

Theorem 4.2 (Backwards Mean-Value Property)**.** *(Evans p. 25) If $u \in C^2(\Omega)$ satisfies*

$$u(x) = \fint_{\partial B(x,r)} u(y)dS = \fint_{B(x,r)} u(y)dy$$

for any open ball $B(x,r) \in \Omega$, then u is harmonic in Ω.

Theorem 4.3 (Regularity of Harmonic Functions)**.** *(Evans p. 28) If $u \in C^0(\Omega)$ satisfies*

$$u(x) = \fint_{\partial B(x,r)} u(y)dS = \fint_{B(x,r)} u(y)dy$$

for any open ball $B(x,r) \in \Omega$, then u is smooth in Ω.

Theorem 4.4 (Mean-Value Bound for Subharmonic Functions)**.** *(Evans p. 85) Let $v \in C^2(\bar{\Omega})$ be subharmonic and then for any $B(x,r) \subset \Omega$,*

$$v(x) \leq \fint_{B(x,r)} v dy$$

5. Maxima Principle

Theorem 5.1 (Weak Maximum Principle)**.** *(Evans p. 27) Let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ be harmonic. Then $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$ and $\min_{\bar{\Omega}} u = \min_{\partial\Omega} u$.*

Theorem 5.2 (Strong Maximum Principle)**.** *(Evans p. 27) Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ be harmonic and assume Ω is connected. If there is an interior point x_0 where u attains its maximum (or minimum) then u is constant.*

Theorem 5.3 (Uniqueness of Dirichlet Problem Solution)**.** *(Evans p. 28) There exists at most one solution to*

$$\begin{cases} -\triangle u = f, x \in \Omega \\ u = g, x \in \partial\Omega \end{cases}$$

where $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ and $f \in C^0(\Omega)$ and $g \in C^0(\partial\Omega)$.

6. Estimates

Theorem 6.1. *(Evans p. 29) Let u be harmonic in Ω, $B(x_0, r) \subset \Omega$, α be a multiindex of order k. Then*

$$|D^\alpha u(x_0)| \leq \frac{C_k}{r^{n+k}} ||u||_{L^1(B(x_0,r))}$$

where $C_0 = \frac{1}{\alpha(n)}$ and $C_k = \frac{(2^{n+1}nk)^k}{\alpha(n)}$ for $k \geq 1$.

Theorem 6.2 (Liouville)**.** *(Evans p. 30) If $u : \mathbb{R}^n \to \mathbb{R}$ is harmonic and bounded, then u is constant.*

Theorem 6.3. *(Evans p .31) If $u : \Omega \to \mathbb{R}$ is harmonic, then it is analytic in Ω.*

Theorem 6.4 (Harnack Inequality)**.** *(Evans p. 32) Let V be a proper open connected subset of Ω and let u be nonnegative and harmonic in u. Then there exists a positive constant C depending on V for which*

$$\sup_V u \leq C \inf_V u$$

and this C is independent of u.

References

[Ev] L.C. Evans, *Partial Differential Equations*, 2nd Edition, American Mathematical Society, 2013.
[Han] Q. Han, *A Basic Course in Partial Differential Equations*, 1st Edition, Americal Mathematical Society, 2011.