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Mathematics

Optimal Design and Asymptotic Compatibility of Nonlocal Problems

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Abstract

This project gives an analytical and numerical treatment of nonlocal optimal design problems where the underlying physical behavior is governed by either nonlocal diffusion or the bond-based model of peridynamics. One objective of this work is to showcase how to recast classical variational theory into the nonlocal setting, with an isotropic class of designs. Beyond that, convergence in the vanishing nonlocal limit and a finite element error analysis are provided. The compliance structure of the cost functional plays an important role in proving these convergence results since it allows us to algebraically relate the state equation to the cost functional.

Notation

- $\Omega \subset \mathbb{R}^n$ be open and bounded, $\Omega_\delta := \Omega + B(0,\delta)$, $\mathcal{D}_\delta := (\Omega_\delta \times \Omega) \cup (\Omega \times \Omega_\delta)$
- $g \in L^2(\Omega)$ is a fixed external force
- Admissible design class (for $0 < a_{\min} < a_{\max}$ fixed):

$$\mathcal{H} \;:=\; \{\mathfrak{a} \in L^\infty(\Omega) \mid \mathfrak{a}(x) \in [a_{\min}, a_{\max}],\ \text{a.e. } x \in \Omega\}$$

- Cost functional:

$$J(\mathfrak{a}, u) \;=\; \int_\Omega g(x)u(x)dx + \frac{1}{2}\|\mathfrak{a}\|^2_{L^2(\Omega)}$$

- Bi-linear forms:

$$B_{\delta,\mathfrak{a}}(u,v) \;:=\; \frac{1}{2}\iint_{\mathcal{D}_\delta} (\mathfrak{a}(x) + \mathfrak{a}(y))\frac{k_\delta(x-y)}{|x-y|^2}(u(x)-u(y))(v(x)-v(y))dxdy$$

$$B_{0,\mathfrak{a}}(u,v) \;:=\; \frac{1}{n}\int_\Omega \mathfrak{a}(x)\nabla u(x)\cdot\nabla v(x)dx$$

- Function spaces:

$$X_0(\Omega_\delta) \;:=\; \{u \in L^2(\Omega_\delta) \mid B_{\delta,\mathfrak{a}}(u,u) < \infty,\ u = 0 \text{ on } \Omega_\delta \setminus \Omega\} \qquad H^1_0(\Omega) \;:=\; \{u \in L^2(\Omega) \mid \nabla u \in L^2(\Omega;\mathbb{R}^n),\ u = 0 \text{ on } \partial\Omega\}$$

Asymptotic Compatibility Diagram

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\text{Find } (\overline{\mathfrak{a}_{\delta,h}}, \overline{u_{\delta,h}}) : \begin{cases} \min\{J(\mathfrak{a}_{\delta,h}, u_{\delta,h})\} \\ (\mathfrak{a}_{\delta,h}, u_{\delta,h}) \in \mathcal{H} \times X_{\delta,h} \\ B_{\delta,\mathfrak{a}_\delta}(u_{\delta,h}, v_{\delta,h}) \;=\; \langle g, v_{\delta,h}\rangle \end{cases} \arrow[r, "h\to 0^+"] \arrow[d, "\delta\to 0^+"] \arrow[dr, "k\to\infty"] & \text{Find } (\overline{\mathfrak{a}_\delta}, \overline{u_\delta}) : \begin{cases} \min\{J(\mathfrak{a}_\delta, u_\delta)\} \\ (\mathfrak{a}_\delta, u_\delta) \in \mathcal{H} \times X_0(\Omega_\delta) \\ B_{\delta,\mathfrak{a}_\delta}(u_\delta, v_\delta) \;=\; \langle g, v_\delta\rangle \end{cases} \arrow[d, "\delta\to 0^+"] \\
\text{Find } (\overline{\mathfrak{a}_h}, \overline{u_h}) : \begin{cases} \min\{J(\mathfrak{a}_h, u_h)\} \\ (\mathfrak{a}_h, u_h) \in \mathcal{H} \times X_h \\ B_{0,\mathfrak{a}}(u_h, v_h) \;=\; \langle g, v_h\rangle \end{cases} \arrow[r, "h\to 0^+"] & \text{Find } (\overline{\mathfrak{a}}, \overline{u}) : \begin{cases} \min\{J(\mathfrak{a}, u)\} \\ (\mathfrak{a}, u) \in \mathcal{H} \times H^1_0(\Omega) \\ B_{0,\mathfrak{a}}(u, v) \;=\; \langle g, v\rangle \end{cases}
\end{tikzcd}$$

Figure 1. Commutative diagram for our family of nonlocal, discrete problems; all convergences displayed are in the strong $L^2(\Omega) \times L^2(\Omega)$ topology

Asymptotic Compatibility Proof Method

Pick positive sequences of parameters $\{\delta_k\}_{k=1}^\infty$ and $\{h_k\}_{k=1}^\infty$ so $\delta_k \to 0$ and $h_k \to 0$. Let $\{(\overline{\mathfrak{a}_k}, \overline{u_k}\}_{k=1}^\infty$ denote a family of solutions to the nonlocal control problem with parameters δ_k and h_k. By compactness there exists $\overline{\mathfrak{a}} \in \mathcal{H}$ so $\overline{\mathfrak{a}_k} \overset{*}{\rightharpoonup} \overline{\mathfrak{a}}$ in the weak-* $L^\infty(\Omega)$ topology. Define $\overline{u}$ as the solution to the local state equation $B_{0,\overline{\mathfrak{a}}}(\overline{u}, v) \;=\; \langle g, v\rangle$ for all $v \in H^1_0(\Omega)$.

Step 1: Prove that $\text{liminf}_{k\to\infty} J(\overline{\mathfrak{a}_k}, \overline{u_k}) \;\geq\; J(\overline{\mathfrak{a}}, \overline{u})$.

Use that $\overline{u_k}$ is a discrete solution to $B_{\delta_k,\overline{\mathfrak{a}_k}}(\overline{u_k}, v_k) \;=\; \langle g, v_k\rangle$, to show that

$$\text{liminf}_{k\to\infty} \int_\Omega g(x)\overline{u_k}(x)dx \;\geq\; \int_\Omega g(x)\overline{u}(x)dx.$$

Step 2: Prove that $\text{limsup}_{k\to\infty} J(\overline{\mathfrak{a}_k}, \overline{u_k}) \;\leq\; J(\overline{\mathfrak{a}}, \overline{u})$.

Let $\widetilde{u_k}$ denote the nonlocal Ritz projection associated with $\overline{\mathfrak{a}}$, then

$$J(\overline{\mathfrak{a}_k}, \overline{u_k}) \;\leq\; J(\overline{\mathfrak{a}}, \widetilde{u_k}).$$

Send $k \to \infty$.

Step 3: Show that $(\overline{\mathfrak{a}}, \overline{u})$ solves the local design problem.

Similar to Step 2.

Step 4: Show that $\overline{u_k} \to \overline{u}$ strongly in $L^2(\Omega)$.

Using Steps 1-2 we have

$$\lim_{k\to\infty} \int_\Omega g(x)\overline{u_k}(x)dx \;=\; \int_\Omega g(x)\overline{u}(x)dx,$$

and then use the state equations to prove

$$\lim_{k\to\infty} \|\overline{u_k} - \overline{u}\|^2_{L^2(\Omega)} \;\lesssim\; \lim_{k\to\infty} B_{\delta_k,\overline{\mathfrak{a}_k}}(\overline{u_k} - \overline{u}, \overline{u_k} - \overline{u}) \;=\; 0.$$

Step 5: Improve coefficient convergence to $\overline{\mathfrak{a}_k} \to \overline{\mathfrak{a}}$ strongly in $L^2(\Omega)$.

Leverage Step 4 and structure of the cost functional.

Important note: The solution $(\overline{\mathfrak{a}}, \overline{u})$ reached depends on which sub-sequence is chosen. Each problem in Figure 1 has a solution but it need not be unique.

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