RESIDUES AND CONTOURS: THEOREMS IN COMPLEX ANALYSIS

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1. Introduction

This document is designed as a reference for theorems in complex analysis (University of Tennessee-Knoxville, MATH 546 Spring 2020) on integration, Cauchy's many theorems of complex analysis, residues, and the associated techniques for evaluating integrals over real and complex domains. Many of these theorems are generalizations of each other. Finally, assume all integrals around circles are oriented counterclockwise for sake of convention.

2. Cauchy Theorems

Theorem 2.1 (Cauchy Integral Formula). *Let f be holomorphic in $|z|< R$ and let $0 < r < R$. Then for $\alpha \in B(0, r)$,*

$$f(\alpha) = \frac{1}{2\pi i}\int_{|z|=r} \frac{f(z)}{z-\alpha}dz.$$

Theorem 2.2 (Goursat). *Let f be holomorphic in a disc $\mathbb{D}$. Then $\int_{\partial T} f dz = 0$ for any triangle T having boundary in $\mathbb{D}$.*

Theorem 2.3 (Cauchy for Convex Regions). *Let $G \subset \mathbb{C}$ be convex (or even star-like) and open, f holomorphic in G. Let γ be a piecewise C^1 closed path in G. Then $\int_\gamma f(z)dz = 0$.*

Theorem 2.4 (Cauchy Theorem for a Disc). *Let $\overline{B(z_0, r)} \subset G$ where G is an open subset of $\mathbb{C}$ and f is holomorphic in G. Then if $\alpha \in B(z_0, r)$ we have*

$$f(\alpha) = \int_{|z-z_0|=r} \frac{f(z)}{z-a} \cdot \frac{dz}{2\pi i}.$$

Theorem 2.5. *Let $G \subset \mathbb{C}$ be open, $f : G \to \mathbb{C}$ be continuous, $\int_{\partial T} f(z)dz = 0$ for all triangles $T \subset G$. Then f is holomorphic.*

Theorem 2.6. *Let $f \in H(A(z_0; R_1, R_2))$, set $C_r := \{|z-z_0|= r\}$. Then $\int_{C_r} f(z0dz$ is independent of $r \in [R_1, R_2]$.*

Theorem 2.7 (Cauchy Integral for Annulus). *Let $f \in H(A(z_0; R_1, R_2))$ with $R_1 < r_1 < r_2 < R_2$. Then for w in $A(z_0, r_1, r_2)$,*

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$$f(w) = \int_{C_{r_2}} \frac{f(z)}{z-w} \frac{dz}{2\pi i} - \int_{C_{r_1}} \frac{f(z)}{z-w} \frac{dz}{2\pi i}.$$

Definition 2.8. *A simple contour is a finite sum of piecewise C^1 closed curves each with their own parameterization.*

Definition 2.9. *Let Γ be a closed contour and z_0 be a point inside the contour. Then the index is ind$_\Gamma(z_0) = \int_\Gamma \frac{1}{z-z_0} \frac{dz}{2\pi i}$.*

Theorem 2.10. *If Γ is a simple contour and $f \in H(G)$ with $\Gamma \subset G$ and G open, then $\int_\Gamma f(z) dz = 0$.*

Further, if $z_0 \in \int \Gamma$ then $f(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z-z_0} dz$.

Theorem 2.11. *The integral of a function with a primitive around a closed path or contour is zero.*

3. Taylor Theorem

Theorem 3.1 (Taylor)**.** *Let $G \subset \mathbb{C}$ be open, $\overline{B(z_0, R)} \subset G$. Then if $f(z) = \int_{|\xi - z_0| = R} \frac{f(\xi)}{\xi - z} \frac{d\xi}{2\pi i}$ we can find a power series expansion with Taylor Coefficients*

$$f^{(n)}(z) = \int_{|\xi - z_0| = R} n! \cdot \frac{f(\xi)}{(\xi - z)^{n+1}} \frac{d\xi}{2\pi i}.$$

4. Residue Theorems

Definition 4.1. *If $f(z) = \sum_{-N}^{\infty} a_n z^n$ with $a_{-N} \neq 0$ then f has a pole of order N at 0 and it has a residue of a_{-1}.*

Definition 4.2. *A rational function R has a removable singularity at ∞ if $R\left(\frac{1}{z}\right)$ has a removable singularity at 0.*

Theorem 4.3 (Residue Theorem)**.** *Let $G \subset \mathbb{C}$ be connected and open. Take a contour $\Gamma \subset G$ homologous to 0. Let $f \in H(G \setminus \{z_1, z_2, ..., z_n\})$ where $z_1, ..., z_n \in G \setminus \Gamma$. Then*

$$\int_\Gamma f(z) \frac{dz}{2\pi i} = \sum_{k=1}^{n} \mathit{ind}_\Gamma(z_k) \mathit{res}_{z_k}(f).$$

Theorem 4.4. *Let $f, g \in H(G)$ with g having a simple zero at $z_0 \in G$. Then*

$$\mathit{res}_{z_0}\left(\frac{f}{g}\right) = \frac{f(z_0)}{g'(z_0)}.$$

If instead g has a zero of order n then

$$\mathit{res}_{z_0}\left(\frac{f}{g}\right) = \frac{1}{(n-1)!} \frac{d^{n-1}}{dz} \left[(z-z_0)^n \frac{f(z)}{g(z)}\right]_{z=z_0}.$$