On the Combinatorics of Placing Balls into Ordered Bins

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Motivating Questions (easy)

- How many ways can we split n balls into ℓ nonempty ordered bins [stars and bars]? $\binom{n+\ell-1}{\ell-1}$

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- How many ways can we split n balls into ℓ nonempty ordered bins [stars and bars]? $\binom{n+\ell-1}{\ell-1}$
- How many ways can we split n balls into ℓ nonempty ordered bins so that each bin has at least t balls?

Motivating Questions (hard)

A similar-sounding, more complicated question:
$B_{n,k}$: How many ways can we split n balls into any number of nonempty ordered bins where the most crowded bin has exactly k balls?
One more restriction:
$M_{n,\ell,k}$: How many ways we can split n balls into ℓ non-empty bins such that the most crowded bin has exactly k balls?

Tools

- Pure enumerative techniques–no generating functions!

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- Pure enumerative techniques–no generating functions!
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- Principle of Inclusion and Exclusion
- Extensive casework

Main Results

Theorem (Closed form: dominant bin)

If $n, k \in \mathbb{N}^+$ with $\frac{n}{2} < k < n$ then

$$B_{n,k} = (n-k+3)2^{n-k-2}.$$

Theorem (Closed form : $B_{2k,k}$)

If $k \in \mathbb{N}^+$ then

$$B_{2k,k} = (k+3)2^{k-2} - 1.$$

Main Results (continued)

Theorem (Closed form for $B_{2k+j,k}$)

Let $j, k \in \mathbb{N}^+$ with $k > j$. Then the number of ways to split $2k + j$ balls into nonempty bins so the most crowded bin has exactly k balls is

$$B_{2k+j,k} \; = \; (k + j + 3)2^{k+j-2} - (3j^2 + 19j + 18)2^{j-4}.$$

Main Results (continued)

Theorem (Formula for Generalized balls into bins with restrictions problem)

Suppose $n, k, \ell \in \mathbb{N}^+$ *such that* $\ell + k - 1 \leq n \leq \ell k$. *Then the following identity for* $M_{n,\ell,k}$ *holds:*

$$M_{n,\ell,k} = \sum_{t=0}^{\ell} (-1)^t \binom{\ell}{t} \left[\binom{n - tk - 1}{\ell - 1} - \binom{n - t(k-1) - 1}{\ell - 1} \right]$$

Lemma

Let $n, \ell, k \in \mathbb{N}^+$ such that $\frac{n}{2} < k < n$ and $2 \leq \ell \leq n - k + 1$. Then the number of ways to split n balls into ℓ nonempty bins where the most crowded bin has exactly k balls is

$$M_{n,\ell,k} = \ell\binom{n-k-1}{\ell-2}.$$

- One bin has k balls
- Split $n - k$ balls amongst $\ell - 1$ nonempty bins; stars and bars
- Decide where the dominant bin goes

Theorem (Closed form: dominant bin)

If $n, k \in \mathbb{N}^+$ with $\frac{n}{2} < k < n$, then

$$B_{n,k} \;=\; (n-k+3)2^{n-k-2}.$$

- Observe $B_{n,k} = \sum_{\ell=2}^{n-k+1} \ell \binom{n-k-1}{\ell-2}$ from the lemma
- Result follows from well-known identity $\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$

Lemma (Formula for $B_{2k,k}$)

If $k \in \mathbb{N}^+$ then the number of ways to split $2k$ balls into ℓ nonempty bins where $2 \leq \ell \leq k+1$ and the most crowded bin has exactly k balls is

$$M_{2k,\ell,k} = \begin{cases} 1, & \ell = 2 \\ \ell\binom{k-1}{\ell-2}, & 3 \leq \ell \leq k+1 \end{cases} .$$

- For case $\ell = 2$, there is only one possible configuration
- For case $\ell \geq 3$, select 1 bin to fill with k (maximum) balls; for the rest of the balls and bins, we use stars and bars
- Maximum number of bins is $k + 1$

Theorem (Closed Form for $B_{2k,k}$)

If $k \in \mathbb{N}^+$ then

$$B_{2k,k} = (k+3)2^{k-2} - 1.$$

- The result is obtained by summing up all possibilities of the previous lemma for the case $2 \leq \ell \leq k+1$

Formula for $B_{2k+j,k}$

In the following slides, we will derive a closed form for $B_{2k+j,k}$ when $0 < j < k$. We introduce some new notation.

- $T_{2k+j,[k+i,k]}$: The number of ways to sort $2k+j$ balls into nonempty bins so that one bin has k balls and another has $k+i$ balls $(0 < i < j)$
- $U_{2k+j,\ell,[k+i,k]}$: The aforementioned quantity where exactly ℓ nonempty bins are used
- $F_{2k+j,k,t}$: The total number of ways to split $2k+j$ balls into nonempty bins so that at least t bins have exactly k balls. In this case $t=1$ or $t=2$
- $G_{2k+j,\ell,k,t}$: The aforementioned quantity where exactly ℓ nonempty bins are used

Lemma

Let $i, j, k \in \mathbb{N}^+$ with $k > j > i$. Then the number of ways to sort $2k + j$ balls into nonempty bins so one bin has k balls and another has $k + i$ balls is

$$T_{2k+j,[k+i,k]} \;=\; \sum_{\ell=1}^{j-i} (\ell^2 + 3\ell + 2) \binom{j-i-1}{\ell-1}.$$

Lemma

Let $j, k \in \mathbb{N}^+$ with $k > j$. Then the total number of ways to split $2k + j$ balls into nonempty bins so that at least two bins have exactly k balls is

$$F_{2k+j,k,2} = \sum_{\ell=1}^{j} \frac{\ell^2 + 3\ell + 2}{2} \binom{j-1}{\ell-1}.$$

Lemma

Let $j, k \in \mathbb{N}^+$ with $k > j$. Then the total number of ways to split $2k + j$ balls into nonempty bins so that at least one bin has exactly k balls is

$$F_{2k+j,k,1} = \sum_{\ell=1}^{k+j} (\ell+1)\binom{k+j-1}{\ell-1} - \sum_{\ell=1}^{j} \frac{\ell^2+3\ell+2}{2}\binom{j-1}{\ell-1}.$$

- We deliberately over-count number of ways to split $2k + j$ balls into nonempty bins so that at least one bin has exactly k balls, grouping $k + j$ balls into ℓ nonempty bins
- We double-count configurations that have two bins with exactly k balls each (ignoring possibility that two bins have k balls)

Lemma

Let $i, j, k, \ell \in \mathbb{N}^+$ with $i < j < k$ and $3 \leq \ell \leq j - i + 2$. Then, the number of ways to split $2k + j$ balls into ℓ nonempty bins so one bin has k balls and another has $k + i$ balls is

$$U_{2k+j,\ell,[k+i,k]} \;=\; \ell(\ell-1)\binom{j-i-1}{\ell-3}.$$

Lemma

Let $j, k, \ell \in \mathbb{N}^+$ *with* $j < k$ *and* $3 \leq \ell \leq j + 2$. *Then the total number of ways to split* $2k + j$ *balls into* ℓ *nonempty bins so that at least two bins have exactly* k *balls is*

$$G_{2k+j,\ell,k,2} \;=\; \frac{\ell(\ell-1)}{2}\binom{j-1}{\ell-3}.$$

Lemma

Let $j, k \in \mathbb{N}^+$ with $j < k$ and $2 \leq \ell \leq k + j + 1$. Then the total number of ways to split $2k + j$ balls into ℓ nonempty bins so the most crowded bin has exactly k balls is

$$M_{2k+j,\ell,k} = \begin{cases} 0, & \ell = 2 \\ \ell\binom{k+j-1}{\ell-2} - \frac{\ell^2-\ell}{2}\binom{j-1}{\ell-3} - \sum_{i=1}^{s}(\ell^2-\ell)\binom{j-i-1}{\ell-3}, & \\ & \ell = j+2-s, 1 \leq s < j \\ \ell\binom{k+j-1}{\ell-2} - \frac{\ell^2-\ell}{2}\binom{j-1}{\ell-3}, & \ell = j+2 \\ \ell\binom{k+j-1}{\ell-2}, & j+3 \leq \ell \leq k+j+1 \end{cases}$$

Frame Title

Lemma (Summation Formula for $B_{2k+j,k}$)

Let $j, k \in \mathbb{N}^+$ with $k > j$. Then the total number of ways to split $2k + j$ balls into nonempty bins so the most crowded bin has exactly k balls is

$$B_{2k+j,k} = \sum_{\ell=2}^{k+j+1} \ell \binom{k+j-1}{\ell-2} - \sum_{\ell=3}^{j+2} \frac{\ell^2-\ell}{2} \binom{j-1}{\ell-3} - \sum_{i=1}^{j-1} \sum_{\ell=3}^{j-i+2} (\ell^2-\ell) \binom{j-i-1}{\ell-3} - 2.$$

Closed Form for $B_{2k+j,k}$

Theorem (Closed Form for $B_{2k+j,k}$)

Let $j, k \in \mathbb{N}^+$ with $k > j$. Then the number of ways to split $2k + j$ balls into nonempty bins so the most crowded bin has exactly k balls is

$$B_{2k+j,k} \;=\; (k+j+3)2^{k+j-2} - (3j^2 + 19j + 18)2^{j-4}.$$

- Comes from finding closed forms of the various summands in the previous lemma
- Used formula $\sum_{k=0}^{n} k^2 \binom{n}{k} \;=\; n(n+1)2^{n-2}$

Generalized Definitions

To generalize the previous formula for n balls, where n is not dependent on the number of bins ℓ, we define:

- n balls instead of $2k + j$ balls
- Maximum capacity is k

Generalized Bins Restriction problem

Definition (Generalized Bins Restriction Problem)

Let $n, \ell, k \in \mathbb{N}^+$. The **Generalized Bins Restriction problem** aims to find the number of ways to split n balls into ℓ bins such that the maximum number of balls in each bin is at most k.

Denote this quantity $R_{n,\ell,k}$

Generalized balls into bins with restrictions problem

Generalized Balls in Bins with restriction

Let $n, \ell, k \in \mathbb{N}^+$. This problem asks how many ways can we split n balls into ℓ bins such that the most crowded bin has exactly k balls?

Denote this quantity $M_{n,\ell,k}$

Formula for generalized bins restriction problem: $R_{n,\ell,k}$

Theorem (Formula for $R_{n,\ell,k}$)

Let $n, \ell, k \in \mathbb{N}^+$. Then the number of ways to split n balls into ℓ bins such that the maximum number of balls in each bin is at most k equals

$$R_{n,\ell,k} \;=\; \sum_{t=0}^{\ell}(-1)^t\binom{\ell}{t}\binom{n-t(k+1)+\ell-1}{\ell-1}.$$

Proof of Formula for $R_{n,\ell,k}$

- Ways to fill n balls into ℓ bins with no upper restriction on the number of bins is simply

$$\binom{n+\ell-1}{\ell-1.}$$

- Note that this would include all the configurations which satisfy the restriction condition that each bin contains at most k and all the configurations which do not satisfy given conditions

Proof of Formula for $R_{n,\ell,k}$

- Assume that the bins have been labelled in some order, indexed from 1 to ℓ. Let A_i denote the finite set of all configurations such that the i^{th} bin contains more than k balls
- The total number of configurations which violate the restriction condition equals

$$\left|\bigcup_{i=1}^{\ell} A_i\right|.$$

Proof of Formula for $R_{n,\ell,k}$

- Using the Principle of Inclusion and Exclusion, this is equivalent to the following formula:

$$\left|\bigcup_{i=1}^{\ell} A_i\right| = \sum_{i=1}^{\ell} |A_i| - \sum_{1\le i<j\le \ell} |A_i \cap A_j| + \sum_{1\le i<j<k\le \ell} |A_i \cap A_j \cap A_k|$$

$$\cdots + (-1)^{\ell-1} |A_1 \cap A_2 \cdots \cap A_\ell| .$$

- Use casework on each term

Proof of Formula for $R_{n,\ell,k}$

- **Terms in First Summation ($\sum_{i=1}^{\ell} |A_i|$):**

$$\sum_{i=1}^{\ell} |A_i| \;=\; \binom{\ell}{1}\binom{n-1(k+1)+\ell-1}{\ell-1}.$$

- **Terms in the Second Summation ($\sum_{1\le i<j\le\ell} |A_i \cap A_j|$):**

$$\sum_{1\le i<j\le\ell} |A_i \cap A_j| \;=\; \binom{\ell}{2}\binom{n-2(k+1)+\ell-1}{\ell-1}.$$

- Proceed similarly for the other summations given in the formula of Inclusion and Exclusion

Generalized balls into bins with restrictions problem

Theorem (Formula for $M_{n,\ell,k}$)

Suppose $n, \ell, k \in \mathbb{N}^+$ such that $\ell + k - 1 \leq n \leq \ell k$. Then the following identity for $M_{n,\ell,k}$ holds:

$$M_{n,\ell,k} = R_{n-\ell,\ell,k-1} - R_{n-\ell,\ell,k-2} = \sum_{t=0}^{\ell}(-1)^t\binom{\ell}{t}\left[\binom{n-tk-1}{\ell-1} - \binom{n-t(k-1)-1}{\ell-1}\right]$$

Proof of formula for $M_{n,\ell,k}$

- Start by putting one ball in each bin to satisfy the "non-emptiness" condition
- Now, the restriction on each bin would be $k - 1$ because we have put one ball in each bin
- Remaining number of balls is $(n - \ell)$, so we fill the bins with the rest of the $(n - \ell)$ balls with the above restriction
- The number of ways to distribute $(n - \ell)$ balls into ℓ bins such that each bin gets at most $k - 1$ balls is simply $R_{n-\ell,\ \ell,\ k-1}$

Proof of formula for $M_{n,\ell,k}$

- There are some unwanted configurations in the above formula; need a bin with **exactly** k balls
- If the most crowded bin does not contain k balls, it contains at most $k-1$ balls
- Distribute n balls into ℓ non-empty bins such that each bin contains at most $k-1$ balls: $R_{n-\ell,\ell,k-2}$
- Subtract the above value from $R_{n-\ell,\ell,k-1}$

Identities involving $R_{n,\ell,k}$

Prominent identities involving the variable $R_{n,\ell,k}$ are highlighted below:

- $$R_{n,\ell,k} = R_{\ell k-n,\ell,k}; \qquad (1)$$

- $$R_{n,\ell,m+k} = \sum_{i=0}^{n} R_{i,\ell,m}\, R_{n-i,\ell,k}; \qquad (2)$$

- $$R_{n,\ell+1,k} = \sum_{i=0}^{k} R_{n-i,\ell,k}; \qquad (3)$$

- $$R_{n+1,\ell+1,k} - R_{n,\ell+1,k} = R_{n+1,\ell,k} - R_{n-k,\ell,k}. \qquad (4)$$

Future Problems

- Find formula for average number of bins, fixing number of balls
- Find formulas when t bins are “tied" for max number of balls
- Color the balls c different colors
- Corresponding probability distributions as $n \to \infty$

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