SIMPLY CONNECTED DOMAINS: THEOREMS IN COMPLEX ANALYSIS

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1. Introduction

This document is designed as a reference for theorems in complex analysis (University of Tennessee-Knoxville, MATH 546 Spring 2020) on simply connected domains. This includes definitions and theorems discussing necessary and sufficient conditions for sets being simply connected, not to mention applications to holomorphic function theory and logarithms. The wording closely follows that of [Sar], but some references are made to [Fis] as well.

2. Different forms of the Definition

Definition 2.1 (Sarason p. 125)**.** *A **domain** is a nonempty, connected, open subset of* $\mathbb{C}$*.*

Definition 2.2 (Sarason p. 126)**.** *A domain* G *in* $\mathbb{C}$ *is **simply connected** if* $\bar{\mathbb{C}} \setminus G$ *is a connected set.*

Definition 2.3 (Fisher p. 107)**.** *A domain* G *in* $\mathbb{C}$ *is **simply connected** if whenever* γ *is a simple closed curve in* G*, the inside of* γ *is also a subset of* G*.*

3. Basic Properties

Lemma 3.1 (Fisher p. 107)**.** *Any convex domain is simply connected.*

4. Equivalence Theorems

Theorem 4.1 (Sarason p. 126)**.** *A domain* G *in* $\mathbb{C}$ *is simply connected if and only if every contour in* G *has winding number* 0 *about every point in* $\mathbb{C} \setminus G$*.*

Theorem 4.2 (Sarason p. 126)**.** *Let* G *be an open subset of* $\mathbb{C}$*. The set* $\bar{\mathbb{C}} \setminus g$ *is connected if and only if every connected component of* G *is simply connected (posed as an exercise)*

Theorem 4.3 (Sarason p. 126)**.** *If* f *is a holomorphic function in the simply connected domain* G *then* $\int_\Gamma f(z)dz = 0$ *for any contour* Γ *in* G*.*

Theorem 4.4 (Sarason p. 127)**.** *A domain* G *in* $\mathbb{C}$ *is simply connected if and only if every holomorphic function in* G *has a primitive.*

Theorem 4.5 (Sarason p. 127)**.** *If a domain* G *in* $\mathbb{C}$ *is simply connected and* f *is a nowhere vanishing holomorphic function in* G*, then there is a branch of the logarithm of* f *in* G*.*

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Theorem 4.6 (Sarason p. 128)**.** *Let G be a simply connected domain in $\mathbb{C}$, and let u be a real-valued harmonic function in G. Then u has a harmonic conjugate in G, unique to within addition of a real constant.*

Definition 4.7 (Sarason p. 128)**.** *A closed curve in G is null homotopic in G if it is homotopic in G to a constant curve, a curve with range being a singleton.*

Theorem 4.8 (Sarason p. 128)**.** *Let G be a domain in $\mathbb{C}$ and let every closed curve in G be null homotopic in G. Then G is simply connected.*

REFERENCES

[Fis] S. Fisher, “Complex Variables," 2nd edition (1990). Dover Publications, Inc.
[Sar] D. Sarason, “Complex Function Theory," 2nd edition (2007). American Mathematical Society.