SOME PROBLEMS AND EXTENSIONS: ANALYSIS AND PDE PRELIM

JOSHUA M. SIKTAR

CONTENTS

Date: August 13, 2020.

1. Introduction

This document is designed as a log, a journal if you will, tracking problems I studied in preparation for the University of Tennessee preliminary examinations in analysis and partial differential equations. I broke them up into rough thematic categories, though the content of the two exams does overlap in a select few areas. Some notes:

(1) Some problems are *based on* old prelim problems, or old homework questions, but ask questions that are different in some way, as to provide more practice.
(2) Some problems naturally fall into more than one category, in which case I only listed them in one category
(3) Hints for some problems are provided at the end
(4) The problems are not arranged by order of difficulty, but the difficulty varies widely between problems (at least in my opinion).

2. Notational Assumptions

(1) $B(x,r)$ denotes an open ball with center x and radius $r > 0$. $\overline{B(x,r)}$ is the closed ball with center x and radius $r > 0$.
(2) Ω is an open, bounded subset of $\mathbb{R}^n$
(3) $\Omega_T = \Omega \times [0,T]$ is a parabolic domain
(4) $(X, \mathcal{A}, \mu)$ denotes a measure space where $\mathcal{A}$ is a sigma-algebra on X and μ a measure; if not stated otherwise take μ to be a positive [real-valued] measure
(5) m denotes Lebesgue measure on the real line
(6) $\alpha(n)$ denotes the volume of a unit ball in $\mathbb{R}^n$; also, $n\alpha(n)$ will be its surface area. Some texts, such as [Han], use ω_n in place of $\alpha(n)$.
(7) $H(D)$ for a set D denotes that a function is [complex] holomorphic in D

3. Automorphisms

3.1. **Problem 1.** Let $a \in \mathbb{D}$ and consider the function $\phi_a(z) = \frac{z-a}{1-\bar{a}z}$. This is called a **disc automorphism**.

a. Show that ϕ maps $\mathbb{D}$ into $\mathbb{D}$.
b. Show that ϕ_a is invertible, and its inverse is ϕ_{-a}.

c. Suppose $f : \mathbb{D} \to \mathbb{D}$ is such that $f(a) = 0$ for some $a \in \mathbb{D}$. Construct a function depending on f for which we can invoke Schwarz's Lemma, and use it to derive a pair of inequalities involving f and f', respectively.

4. Characteristics and First-Order PDE

4.1. **Problem 1.** If u is constant along a C^1 characteristic curve $((x(t), y(t))$ in $\mathbb{R}^2$ show that

$$\frac{dx}{dt}\frac{\partial u}{\partial x} + \frac{dy}{dt}\frac{\partial u}{\partial y} = 0$$

How is this related to how we derive the characteristic differential equations corresponding to a linear first-order PDE?

4.2. **Problem 2.** a. Show that $U(x, y, u) = \frac{1}{2}x^2 + \frac{1}{y}$ is a conserved quantity of the PDE $xu_x + y^2 u_y = u$ in $\{(x, y) \in \mathbb{R}^2, y > 0\}$.

b. Show that $U(x, y, u) = \frac{x}{u}$ is a conserved quantity of the PDE $xu_x + y^2 u_y = u$ assuming a priori that $u > 0$.

c. Show that $U(x, y, u) = xe^{-\frac{1}{2}}u^2$ is a conserved quantity of the PDE $xu_x + u_y = 1$.

d. Show that $U(x, y, u) = u - y$ is a conserved quantity of the PDE $xu_x + u_y = 1$.

4.3. **Problem 3.** Consider the following PDE in $\mathbb{R}^2$

$$xyu_x + (x + 1)u_y = u$$

Where are the following hypersurfaces characteristic, if anywhere?

a. $\{y = 0\}$

b. $\{x = y\}$

c. $\{x = -y\}$

4.4. **Problem 4.** Consider the following PDE on $\mathbb{R}^2$:

$$2yu_x + u_y = u^4$$

with data $u(x, 0) = f(x)$. Assume $f \in C^0(\mathbb{R})$.

a. Show that the PDE is noncharacteristic on $\Sigma := \{y = 0\}$.

b. Find a different hypersurface on which this PDE is *characteristic* at the origin (ignore the initial condition here).

c. Identify the compatibility condition for the given PDE in terms of the initial data f.

4.5. **Problem 5.** Consider the following PDE:

$$2xyu_x + u_y = u^4$$

Assume there is initial data $f \in C^2(\mathbb{R})$ for which $u(x, 0) = f(x)$ for $x \in \mathbb{R}$.

a. Show that if $f \geq 0$ on all of $\mathbb{R}$, then a solution exists for all $x \in \mathbb{R}, y < 0$

b. Show that if $f \leq 0$ on all of $\mathbb{R}$, then a solution exists for all $x \in \mathbb{R}, y > 0$

c. Show that if a solution exists for all $x \in \mathbb{R}, y > 0$ (or for all $x \in \mathbb{R}, y < 0$), then f must either be a constant or a linear function.

4.6. **Problem 6.** Let T be the triangle bounded by the lines $y = x, y = 0$, and $y = 2-x$. Consider the PDE

$$xu_x + 2yu_y = u^2$$

with boundary conditions

$$u(x,x) = f(x), x \in (0,1)$$
$$u(x, 2-x) = x^2, x \in (1,2)$$
$$u(x,0) = g(x), x \in (0,2)$$

Find conditions on f and g that ensure the solution is continuous on T, or prove that no such conditions exist.

4.7. **Problem 7.** Let the initial hyper-surface be $\Sigma := \{y = 2x\} \subset \mathbb{R}^2$. Find all functions $f \in C^1(\mathbb{R})$ such that the PDE $u_x + 2u_y = u^2$ satisfies the compatibility condition with $u = f(x)$ on Σ. For each such solution, on what subset of $\mathbb{R}^2$ does it exist?

5. Classification of Second-Order Linear PDE

5.1. **Problem 1.** For each PDE listed, write the principal part and determine whether the PDE is elliptic, hyperbolic, or parabolic.
a. Laplace's Equation in $\mathbb{R}^2$
b. Heat Equation in $\mathbb{R} \times \mathbb{R}$
c. Wave Equation in $\mathbb{R} \times \mathbb{R}$

5.2. **Problem 2.** Let A denote the subset of $\mathbb{R}^2$ in which a given PDE is elliptic. Explicitly construct a second-order linear PDE for which the set A is *not* convex. Note that the empty set is vacuously convex, so A cannot be empty in your example.

5.3. **Problem 3.** Recall that we classify second-order linear PDEs pointwise; that is, the classification can change in different subsets of $\mathbb{R}^n$.
a. Show that if the principal part of the operator representing the PDE has constant coefficients, then the classification of the PDE is the same in all of $\mathbb{R}^n$.
b. Explicitly construct a second-order linear PDE with the same classification (elliptic, hyperbolic, parabolic) in $\mathbb{R}^2$ that does not have all constant coefficients.

5.4. **Problem 4.** Consider the second-order partial differential operator in $\{(x,y) : x, y > 0\} \subset \mathbb{R}^2$:

$$Lu = (\alpha_1 x + \beta_1 y)u_{xx} + (2\alpha_2 x + 2\beta_2 y)u_{xy} + (\alpha_1 x + \beta_1 y)u_{xy}$$

Devise a criterion (a set of inequalities comparing $\alpha_1, \alpha_2, \beta_1, \beta_2$) which ensures the PDE is elliptic in all of $\{(x,y) : x, y > 0\} \subset \mathbb{R}^2$.

6. Conformal Maps

These are Carl Sundberg's favorite. As he did, I encourage you to draw pictures for yourself. Lots of them.

6.1. **Problem 1.** a. Find a sequence of conformal maps between the quarter plane $\{z \in \mathbb{C}, \Re z > 0, \Im z > 0\}$ and the upper half plane $\{z \in \mathbb{C}, \Im z > 0\}$.

b. [January 2011 UTK Analysis Prelim Problem 3] Find a sequence of conformal maps between the upper half plane $\{z \in \mathbb{C}, \Im z > 0\}$ and $\mathbb{D}$.

c. Find a sequence of conformal maps between the lower half plane $\{z \in \mathbb{C}, \Im z < 0\}$ and $\mathbb{D}$.

d. Find a sequence of conformal maps between the left half plane $\{z \in \mathbb{C}, \Re z < 0\}$ and $\mathbb{D}$.

e. Find a sequence of conformal maps between the right half plane $\{z \in \mathbb{C}, \Re z > 0\}$ and $\mathbb{D}$.

f. [August 2011 UTK Analysis Prelim Problem 4] Find a sequence of conformal maps between the unbounded sector $\Omega := \{z \in \mathbb{C}, |z| > 1, 0 < \operatorname{Arg}(z) < \frac{\pi}{3}\}$ and $\mathbb{D}$.

g. [January 2012 UTK Analysis Prelim Problem 1] Find a sequence of conformal maps between the slitted disc $\mathbb{D} \setminus [0, 1)$ and $\mathbb{D}$.

h. Find a sequence of conformal maps between the quarter plane $\{z \in \mathbb{C}, \Re z > 0, \Im z > 0\}$ and the right half plane $\{z \in \mathbb{C}, \Re z > 0\}$.

i. Find a sequence of conformal maps between the strip $\Omega := \{z \in \mathbb{C}, -1 < \Im z < 1\}$ and $\mathbb{D}$.

j. [August 2012 UTK Analysis Prelim Problem 5] Find a sequence of conformal maps between the strip $\Omega := \{z \in \mathbb{C}, 0 < \Re z < 1\}$ and $\mathbb{D}$.

k. Find a sequence of conformal maps between the right half-plane $\{z \in \mathbb{C}, \Re z > 0\}$ and the set $\mathbb{C} \setminus \{z \in \mathbb{R}, z \leq 0\}$.

l. Find a sequence of conformal maps between the "bulb" $\{z \in \mathbb{C}, \Im z < 0\} \cup (\mathbb{D} \cap \{\Im z > 0\})$ and $\mathbb{D}$.

m. Find a sequence of conformal maps between $\mathbb{D} \setminus [0.5, 1)$ and $\mathbb{C} \setminus [0, 1]$.

6.2. **Problem 2.** [Found on Math StackExchange] Provide a contradiction to proof to show that

$$A := \mathbb{D} \setminus \{0\}$$

$$B := \{z \in \mathbb{C}, 1 < |z| < 2\}$$

are not conformally equivalent.

7. Derivative Inequalities for Holomorphic Functions and the Identity Theorem

7.1. **Problem 1.** State and prove Liouville's Theorem for complex-valued functions.

7.2. **Problem 2.** a. Suppose $f : \mathbb{D} \to \mathbb{D}$ is holomorphic, where $f(0) = 0$ and $f'(0) = 0$. Prove that $|f(z)| \leq |z|^2$ in all of $\mathbb{D}$.

b. Additionally assume $f' : \mathbb{D} \to \mathbb{D}$ in addition to the assumptions in part a. Prove that $|f'(z)| \leq |z|$ and $|f''(0)| \leq 1$.

7.3. **Problem 3.** Show there does not exist a function $f : \mathbb{D} \to \mathbb{C}$ holomorphic such that

$$f\left(\frac{1}{n}\right) = \frac{1}{n^2}$$

for $n \in \mathbb{N}^+$ that are multiples of 4 and

$$f\left(\frac{1}{n}\right) = \frac{-1}{n^2}$$

for $n \in \mathbb{N}^+ \setminus \{1\}$ that are not multiples of 4.

7.4. **Problem 4.** a. Suppose $f : \mathbb{D} \to \mathbb{C}$ is holomorphic and

$$f\left(\frac{1}{n}\right) = \frac{n^2}{1+2n^2} + \frac{n}{1+2n}$$

for $n \in \mathbb{N}^+ \setminus \{1\}$. Prove that

$$f(z) = \frac{1}{z^2+2} + \frac{1}{z+2}$$

in $\mathbb{D}$.

b. Why can this result not be extended to the entire complex plane? That is, if f is actually entire with the same restrictions otherwise, what happens?

7.5. **Problem 5.** Suppose f is entire and has real and imaginary parts that are identical to each other in $\mathbb{D}$. Prove that $(i+1)f$ is purely imaginary, and $(i-1)f$ is purely real.

7.6. **Problem 6.** Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic with zeros at 0 and w.

a. Prove that if either zero is not isolated then f must identically be equal to 0.

b. Assuming that both zeros are isolated, define $g(z) := \frac{f(z)}{z(z-w)}$ on $\mathbb{D}$. Show g is itself holomorphic on $\mathbb{D}$.

c. Show that if there exists a point z_0 in $\mathbb{D}$ where $|g(z_0)| > \frac{1}{|w||1-w|}$, then g is a constant.

7.7. **Problem 7.** Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic, let $w \in \mathbb{D} \setminus \{0\}$. Suppose f has a zero of order m_1 at 0 and a zero of order m_2 at w, where $1 \leq m_1, m_2 < \infty$.

a. Prove that for any $z \in \mathbb{D}$,

$$|f(z)| \leq |z|^{\min 1-m_1+m_2, 1-m_2+m_1}$$

b. For which functions f is this bound no better than the one provided by Schwarz's Lemma?

7.8. **Problem 8.** Let $f : \mathbb{D} \to \mathbb{C}$ be holomorphic. Suppose $\Re f > 0$ on all of $\mathbb{D}$.

a. Show that $\Re f'(0) = 0$.

b. [UTK Analysis Prelim January 2016 Problem 5] Show that $|f'(0)| \leq 2\Re f(0)$.

7.9. **Problem 9.** Suppose $f : \bar{\mathbb{D}} \to \mathbb{C}$ is holomorphic in $\bar{\mathbb{D}}$. Also suppose f is real-valued on $\partial\mathbb{D}$. Show that f is constant on $\mathbb{D}$.

7.10. **Problem 10.** Let $f : E \to \mathbb{C}$ be holomorphic, where E is an open subset of $\mathbb{C}$. Prove that f is harmonic in E.

7.11. **Problem 11.** Suppose $u : \mathbb{D} \to \mathbb{R}$ is a real-valued nonnegative function such that $u(x, y) = (x^2 + y^2 + 1)v(x, y)$ is harmonic in $\mathbb{D}$. Prove that

$$u(0) \geq \frac{1}{8\pi}\left(\int_0^{2\pi} \sqrt{|u(e^{i\theta})|d\theta}\right)^2$$

7.12. **Problem 12.** Suppose $f : \mathbb{D} \to \mathbb{D}$ is holomorphic. Let $f\left(\frac{1}{2}\right) = 0$.

a. [UTK Analysis Prelim August 2013 Problem 6] Find the maximum possible value of $|f(0)|$ and find the functions f attaining this value. Justify your answer.

b. [My variant] Prove that for any f satisfying these conditions, that $\left|f'\left(\frac{1}{2}\right)\right| \leq \frac{4}{3}$.

7.13. **Problem 13.** Suppose $w \in \mathbb{D} \setminus \{0\}$ and $f : \mathbb{D} \to \mathbb{D}$ is analytic with $f(0) = f(w) = 0$.

a. [UTK Analysis Prelim August 2016 Problem 6] Prove $|f'(0)| \leq |w|$.

b. [UTK Analysis Prelim August 2016 Problem 6] Characterize all possible functions f where $|f'(0)| = |w|$.

c. [My variant] Prove $|f'(w)| \leq \frac{1}{1-|w|^2}$.

d. [My variant] Prove $|f(w)| \leq |w|$.

e. [My variant] Prove $|f'(0)| \leq 1 - |w|^2$.

8. Differentiation in the Lebesgue Sense

8.1. **Problem 1.** Let $f : [0, 1] \to \mathbb{R}$ be absolutely continuous on $[0, 1]$, and let g be a function mapping $[0, 1]$ onto the positive real numbers. Show that $h(x) := f(x)e^{-g(x)}$ must also be absolutely continuous on $[0, 1]$.

8.2. **Problem 2.** Some sections of this problem were proposed by Carl Sundberg in MATH 545.

Throughout, let $f, g : [a, b] \to \mathbb{R}$ be absolutely continuous.

a. Show that the sum of any two absolutely continuous functions is also absolutely continuous.

b. Show that the difference of any two absolutely continuous functions is also absolutely continuous.

c. Show that the square of any absolutely continuous function is absolutely continuous.

d. From parts a-c deduce that fg must be absolutely continuous.

e. Deduce the integration by parts formula

$$\int_a^b f(x)g'(x)dm = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)dm$$

8.3. **Problem 3.** Some sections of this problem were proposed by Carl Sundberg in MATH 545.

a. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $|f(x) - f(y)| \leq C|x - y|^\alpha$ for all $x, y \in \mathbb{R}$ where $\alpha > 1, C > 0$. Show that f is a constant. This is called Hölder continuity with exponent α.

b. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $|f(x) - f(y)| \leq C|x - y|$ for all $x, y \in \mathbb{R}$ where $C > 0$. Show that f' is bounded.

c. For the bound obtained in part b, give an example of a function where the bound on f' is sharp (i.e. where the obtained nonstrict inequality is an equality)

d. Verify that the result of part a still holds for an entire function $f : \mathbb{C} \to \mathbb{C}$.

e. Let $f : [0,1] \to \mathbb{R}$ be such that $|f(x) - f(y)| \leq C|x-y|$ for all $x, y \in [0,1]$ where $C > 0$.

f. Show that if $f : [\ell, h] \to \mathbb{R}$ is absolutely continuous, then for any $\alpha > 0$ the map $g : [\alpha\ell, \alpha h] \to \mathbb{R}$ defined as $g(x) := f(\alpha x)$ is itself absolutely continuous on $[\alpha\ell, \alpha h]$

g. Show that if $f : [\ell, h] \to \mathbb{R}$ is absolutely continuous, then the map $g : [\ell+t, h+t] \to \mathbb{R}$ defined as $g(x) := f(x+t)$ is itself absolutely continuous on $[\ell+t, h+t]$ for each $t \in \mathbb{R}$.

h. Use parts f and g to conclude that part e extends to any bounded interval $[a,b] \subset \mathbb{R}$.

i. Suppose $f : [a,b] \to \mathbb{R}$ is absolutely continuous and let $p > 1$. Show that $|f|^p$ is absolutely continuous.

j. Using part i, show that for f and p satisfying the conditions of part i,

$$\frac{d}{dx}|f|^p = pf^{p-1}\text{sign}(f)f'$$

8.4. **Problem 4.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous. Show that f is locally integrable in L^p for any $1 \leq p < \infty$.

8.5. **Problem 5.** This problem was a homework problem provided by Carl Sundberg.

a. Let $(X, \mathcal{A}, \mu)$ be a measure space, $\phi : [0,\infty) \to [0,\infty)$ be absolutely continuous with $\phi(0) = 0$, $f \in L^1(\mu)$. Prove that

$$\int \phi \circ f d\mu = \int_0^\infty \phi'(\lambda)\mu([|f| > \lambda])d\lambda$$

b. Let $1 < p < \infty$, with $f(1) = 0$ and f be absolutely continuous on $[1, N]$ for all $N > 1$. Suppose that $f' \in L^p([1,\infty))$. Show that

$$\int_1^\infty \frac{|f(x)|^p}{x^p} dx < \infty$$

9. Elliptic PDE

9.1. **Problem 1.** a. [Henry Simpson Lecture] Let $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ solve

$$\begin{cases} \triangle u = u^3, x \in \Omega \\ u = 0, x \in \partial\Omega. \end{cases}$$

Notice that the zero function is clearly a solution. Prove that the zero function is the only solution via a Maximum Principle.

b. [Henry Simpson Lecture] Prove that the zero function is the only solution via an energy method.

c. [Henry Simpson Lecture] Prove the zero function is the only solution by the following method: let $M := \sup_{\bar{\Omega}} u$, and assume for sake of contradiction $M \geq 0$. Break into cases based on the sign of M.

d. [My extension] Now suppose $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ solves

$$\begin{cases} \triangle u = u^3 e^u, x \in \Omega \\ u = 0, x \in \partial\Omega. \end{cases}$$

By modifying one of the methods from parts a-c, show that once again the zero function is the only solution.

e. [My extension] Here the geometry of Ω comes into play. Let $u \in C^2(B(0,1)) \cap C^0(\partial B(0,1))$ solve

$$\begin{cases} \triangle u = u^3, x \in B(0,1) \\ u = e^{x_1}, x \in \partial B(0,1). \end{cases}$$

Show that $|u| \leq e$ on all of $\overline{B(0,1)}$.

9.2. **Problem 2.** Let $w \in C^1(\Omega) \cap C^0(\bar{\Omega})$ and define the energy

$$E[w] := \frac{1}{2}\int_{\Omega} |\triangledown w|^2 dx - \int_{\partial\Omega} hw dS$$

a. Show that if $u \in C^1(\Omega) \cap C^0(\bar{\Omega})$ solves the Robin Boundary Problem for Laplace's Equation,

$$\begin{cases} \triangle u = 0, x \in \Omega \\ \frac{\partial u}{\partial n} + \alpha(x)u = h(x), x \in \partial\Omega \end{cases}$$

then $E[u] \leq E[w]$. We say that u minimizes the energy functional $E[\cdot]$.

b. Show that if u solves the aforementioned Boundary Value Problem then $E[w] \geq E[u]$ for any $w \in C^1(\Omega) \cap C^0(\bar{\Omega})$.

9.3. **Problem 3.** Let $f, c \in C^0(\bar{\Omega})$ and $g \in C^0(\partial\Omega)$, and let c be positive in $\bar{\Omega}$. Prove there is at most one C^2 solution to the PDE

$$\begin{cases} -e^{x^2+5x+1}u_{xx} - e^{y^2+5y+1}u_{yy} + c(x,y)u = f, (x,y) \in \Omega \\ u = g, \qquad (x,y) \in \partial\Omega \end{cases}$$

9.4. **Problem 4.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $u, h \in C^2(\Omega) \cap C^0(\bar{\Omega})$ such that $u > 0$ in Ω and $a > 0$ is a constant, where

$$\begin{cases} \triangle u = \frac{1}{u} + u^5, x \in \Omega \\ u = a, \qquad x \in \partial\Omega \end{cases}$$

$$\begin{cases} \triangle h = 1, x \in \Omega \\ h = -\frac{a}{2}, x \in \partial\Omega \end{cases}$$

a. Prove that $u \leq a$ in Ω.

b. Prove that $h \leq -\frac{u}{2}$ in Ω.

9.5. **Problem 5.** a. [UTK PDE Prelim Exam August 2015 Problem 2] Let $\Omega \subset \mathbb{R}^2$ be open and bounded, and suppose $g \in C^0(\partial\Omega)$. Show there exists at most one solution $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfying

$$\begin{cases} \triangle u + u_x - u_y = u^3, x \in \Omega \\ u = g, \qquad\qquad\qquad x \in \partial\Omega \end{cases}$$

Use a maximum principle.

b. If we further assume that $\partial\Omega$ is a C^1 boundary, we can also show uniqueness using an energy method. Now show the energy approach.

9.6. **Problem 6.** Let Ω be a bounded domain in $\mathbb{R}^n$, let c be a continuous function in Ω, and $u \in C^2(\Omega) \cap C(\bar{\Omega})$.

a. [A theorem in Han] Prove that if $c \leq 0$ and $\triangle u + cu \geq 0$ in Ω, then $\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u^+$.

b. Prove that if $c \geq 0$ and $\triangle u + cu^2 > 0$ in Ω then $\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u^+$.

c. Using the same conditions as part b, can we also deduce $\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u$?

d. Suppose that $c \leq 0$, $u \geq 0$, and $\triangle u + cu^2 \geq 0$ in Ω. Show then that $\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u^+$.

9.7. **Problem 7.** Suppose $u \in C^2(B(0,1)) \cap C^0(\overline{B(0,1)})$ such that $\triangle u + cu \geq 0$ in $B(0,1)$, for a continuous function $c \leq 0$ in $B(0,1)$. Assume $\exists w \in B(0,1)$ such that $u(w) > 0$.

a. Prove that $\exists \partial B(0,1)$ such that $u(x_0) > 0$.

b. Suppose $\exists 0 < r < 1$ such that $u \leq 0$ on $B(0,1) \setminus B(0,r)$. Show then that $\frac{\partial u}{\partial v}(x_0) > 0$.

9.8. **Problem 8.** Suppose $\Omega \subset \mathbb{R}^n$ is a bounded C^1 domain that satisfies the interior sphere condition. Suppose $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies $u \geq 1$ in Ω, $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$, and $\triangle u + u^3 - u^4 \geq 0$ in Ω. Prove that for each $x_0 \in \partial\Omega$, there exists $x_1 \in \Omega$ such that $u(x_0) \leq u(x_1)$. Deduce that

$$\sup_{\partial\Omega} u = \sup_{\bar{\Omega}} u$$

9.9. **Problem 9.** [UTK PDE Prelim August 2017 Problem 3] Let $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a $C^\infty-$smooth boundary. Suppose p, q are nonnegative continuous functions on Ω such that $p(x) + q(x) > 0$ (strict inequality) for all $x \in \Omega$. Find all functions $u \in C^2(\bar{\Omega})$ satisfying

$$\begin{cases} \triangle u = pu^3 + qu, \qquad x \in \Omega \\ \frac{\partial u}{\partial n} = 0, \qquad\qquad x \in \partial\Omega \end{cases}$$

Try both methods.

a. Use an energy method.

b. Use a maximum principle-type argument along with the Hopf Lemma (proceed by contradiction).

9.10. **Problem 10.** [Tuoc Phan] Let $n \geq 2$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a $C^\infty-$smooth boundary. Suppose p, q are nonnegative continuous functions on Ω such that $p(x)+q(x) > 0$ (strict inequality) for all $x \in \Omega$. Suppose $a(x) > 0$ and $b(x) \geq 0$ on $\partial\Omega$ are continuous as well. Find all functions $u \in C^2(\bar{\Omega})$ satisfying

$$\begin{cases} \triangle u = pu^3 + qu, & x \in \Omega \\ a(x)\frac{\partial u}{\partial n} + b(x)u = 0, & x \in \partial\Omega \end{cases}$$

Try both methods.

a. Use an energy method.

b. Use a maximum principle-type argument along with the Hopf Lemma (proceed by contradiction).

c. Identity a slightly weaker condition on a and b such that the argument from part b will still work.

9.11. **Problem 11.** Let $\Omega := \{x \in \mathbb{R}^3 : 0 < |x| < 1\}$ and $S := \{x \in \mathbb{R}^3, |x| = 1\}$. Suppose $u \in C^2(\Omega) \cap C^0(\Omega \cup S)$ satisfies

$$\begin{cases} \triangle u \geq 0, x \in \Omega \\ u = 0, \quad x \in S \end{cases}$$

with u bounded on Ω.

a. [UTK PDE Prelim Fall 2012 Problem 2] Prove that $u \leq 0$ on Ω by using the auxiliary function

$$v(x) = u(x) - \epsilon\left(\frac{1}{|x|} - 1\right)$$

for $\epsilon > 0$.

b. [My variant] I then tried looking for other auxiliary functions that might give a better bound, for instance I tried using

$$v(x) = u(x) - \epsilon\left(\frac{1}{|x|^2} - 1\right)$$

However, I realized that under the conditions given in the problem, there was no sense in trying to find a better bound. Why is this the case?

10. Entire Functions

10.1. **Problem 1.** a. Suppose f is an entire function with a real part that never vanishes. Prove that f must be constant.

b. [UTK Analysis Prelim January 2006 Problem 4, UTK Analysis Prelim August 2003 Problem 2] Let f be an entire function which satisfies $|f(z)| \leq 1 + \sqrt{|z|}$ for all $z \in \mathbb{C}$. Show that f is a constant.

10.2. **Problem 2.** Give two justifications for why $f(z) = \sin(\Re z)$ is not entire:

a. One using Cauchy-Riemann Equations

b. One using Liouville's Theorem

10.3. **Problem 3.** a. [UTK Analysis Prelim August 2015 Problem 1] A complex valued function f on $\mathbb{C}$ is called locally M-Lipschitz for some $M > 0$ if $\forall z \in \mathbb{C}$ there exists $\epsilon > 0$ so that $|f(w) - f(z)| < M|z - w|$ for all $|w - z| < \epsilon$. Given $M > 0$ state and prove a description of all entire functions f which are locally M-Lipschitz on $\mathbb{C}$.

b. [My variant] Suppose instead f is entire and that for any $z \in \mathbb{C}$, there exists $\epsilon > 0$ such that whenever $|w - z| < \epsilon$,

$$|f(z)^2 - f(w)^2| < M|w - z|$$

Prove that either:

i) f must be a constant

ii) There exist $\alpha, \beta \in \mathbb{C}$ and a g entire and nonzero where $f = \alpha g$ and $f' = \frac{\beta}{g}$

iii) There exist $\alpha, \beta \in \mathbb{C}$ and a g entire and nonzero where $f' = \alpha g$ and $f = \frac{\beta}{g}$

c. Are conditions ii) and iii) equivalent to each other?

10.4. **Problem 4.** a. Prove the Casorati-Weierstrass Theorem: let f be holomorphic in a punctured disc in $D_r(z_0) \setminus \{z_0\}$ with an essential singularity at z_0. Then $f(D_r(z_0) \setminus z_0)$ is dense in $\mathbb{C}$.

b. Provide an alternative proof if we replace the punctured disc with the punctured plane $\mathbb{C} \setminus \{z_0\}$.

11. Fractional Linear Transformations

11.1. **Problem 1.** [Carl Sundberg] Let $f \in H(\mathbb{D})$, $f : \mathbb{D} \to \mathbb{D}$ be invertible. Then there exists an $\alpha \in \mathbb{D}, \lambda \in \partial\mathbb{D}$ such that

$$f(z) = \lambda \frac{z - \alpha}{1 - \bar{\alpha} z}$$

12. Fubini-Tonelli and Product Measures

12.1. **Problem 1.** [Remus Nicoara] Calculate

$$\int_0^1 \int_y^1 x^{-\frac{3}{2}} \cos\left(\frac{\pi y}{2x}\right) dxdy$$

12.2. **Problem 2.** [Remus Nicoara] Let $f(x,y) := e^{-xy} - 2e^{-2xy}$. Explain why the following observation does not contradict Fubini's Theorem:

$$\int_0^1 \int_1^\infty f(x,y)dxdy \neq \int_1^\infty \int_0^1 f(x,y)dydx$$

12.3. **Problem 3.** a. [Remus Nicoara] Let $f : \mathbb{R} \to [0, \infty)$ be Lebesgue measurable. Let $E_t := \{x \in \mathbb{R}, f(x) \geq t\}$ for any $t \geq 0$. Show that

$$\int_{\mathbb{R}} f(x)dx = \int_0^\infty m(E_t)dt$$

b. [Remus Nicoara] Let $A, B \subset \mathbb{R}$ be Lebesgue measurable. Let $H(x) := m((A - x) \cap B)$ for all $x \in \mathbb{R}$. Show that H is a measurable function and that

$$\int_{\mathbb{R}} H(x)dx = m(A)m(B)$$

12.4. **Problem 4.** [Remus Nicoara] Let $f(x,y) := \frac{x^2-y^2}{(x^2+y^2)^2}$ when $(x,y) \neq (0,0)$ and $f(0,0) = 0$. Show that

$$\int_0^1 \int_0^1 f(x,y)dydx \neq \int_0^1 \int_0^1 f(x,y)dxdy$$

and argue why this does not contradict Fubini-Tonelli.

12.5. **Problem 5.** a. [Remus Nicoara] Let f, g be integrable functions on a measure space $(X, \mathcal{M}, \mu)$ where X is σ-finite. For each $t \in \mathbb{R}$ let

$$F_t := \{x, f(x) > t\}$$
$$G_t := \{x, g(x) > t\}$$

Show that

$$\int_{\mathbb{R}} \mu(F_t \triangle G_t)dt = \int_X |f - g| d\mu$$

b. [Remus Nicoara] Show that the claim from part a still holds even if X is not σ-finite.

12.6. **Problem 6.** Carefully and rigorously prove that

$$\int_0^1 \int_0^1 xy^2 - x^2ydxdy = 0$$

12.7. **Problem 7.** Suppose $f : [0,1] \times [0,1] \to \mathbb{R}$ is $m \times m$-measurable and $g : [0,1] \to \mathbb{R}$ is m-measurable. Suppose $g \in L^1(0,1)$ and $\sup_{y \in (0,1)} ||f(\cdot,y)||_{L^2(0,1)} < \infty$. Prove that

$$\int_0^1 \int_0^1 f(x,y)g(y)dydx = \int_0^1 g(y) \int_0^1 f(x,y)dxdy$$

13. Hyperbolic PDE

13.1. **Problem 1.** a. [UTK PDE Prelim August 2012 Problem 4] Let $a, b \in \mathbb{R}, T > 0$. Suppose $\phi, \psi \in C^\infty(\bar{\Omega})$, $u \in C^2(\Omega_T) \cap C^0(\bar{\Omega_T})$ solves

$$\begin{cases} u_{tt} - \triangle u + au_{x_1} + bu = 0, (x,t) \in \Omega_T \\ u = 0, (x,t) \in \partial\Omega \times (0,T] \\ u = \phi, (x,t) \in \Omega \times \{t = 0\} \\ u_t = \psi, (x,t) \in \Omega \times \{t = 0\} \end{cases}.$$

Let $E(t) := \frac{1}{2} \int_\Omega u_t^2 + |\triangledown u|^2 dx$. Prove $\exists k > 0$ such that $E(t) \leq E(0)e^{kt}$ for all $t \in [0,T]$.
b. [My extension] Using part a, derive the estimate

$$||u_t||^2_{L^2(\Omega_T)}+||\bigtriangledown u||^2_{L^2(\Omega_T)}\leq \frac{1}{k}(e^{kT}-1)(||\psi||^2_{L^2(\Omega)}+||\bigtriangledown\phi||^2_{L^2(\Omega)}),$$

and use this to conclude a continuous dependence statement on the initial data of the PDE.

13.2. **Problem 2.** Consider the following BVP for the wave equation. Let $u \in C^2(\mathbb{R}^n \times (0,\infty)) \cap C^0(\mathbb{R}^n \times [0,\infty))$ solve

$$\begin{cases} u_{tt} = \triangle u, (x,t) \in \mathbb{R}^n \times \{t = 0\} \\ u(x,0) = f(x), x \in \mathbb{R}^n \\ u_t(x,0) = g(x), x \in \mathbb{R}^n \end{cases}$$

and suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is harmonic, $g : \mathbb{R}^n \to \mathbb{R}$ is smooth.

a. Suppose first that $n = 1$. Prove that there exist constants $c_1, c_2 \in \mathbb{R}$ such that

$$|u(x,t)|\leq |2c_1x + c_2|+\frac{1}{2}\int_{x-t}^{x+t}|g(s)|ds.$$

for any $x \in \mathbb{R}, t > 0$.

b. Still supposing that $n = 1$, prove there exists $c_1 \in \mathbb{R}$ so the following hold for $x \in \mathbb{R}, t > 0$:

$$|u_x(x,t)|\leq |c_1|+\frac{1}{2}|g(x+t)|+\frac{1}{2}|g(x-t)|$$

$$|u_t(x,t)|\leq \frac{1}{2}|g(x+t)|+\frac{1}{2}|g(x-t)|$$

c. Suppose now that $n = 2$. Prove that for any $0 < t < 1, x \in \mathbb{R}^2$,

$$|u(x,t)|\leq \frac{1}{2\pi t^{\frac{3}{2}}}\int_{B(x,t)}\frac{|f(y)|}{|y-x|}dy + \frac{1}{2\pi\sqrt{t}}\int_{B(x,t)}\frac{|g(y)|}{|y-x|}dy = \frac{1}{2\pi\sqrt{t}}\int_{B(x,t)}|\bigtriangledown f(y)|dy.$$

d. Still supposing that $n = 2$, prove that for any $t > 0, x \in \mathbb{R}^2$,

$$|u(x,t)|\leq \frac{1}{2\pi t^{\frac{3}{2}}}\int_{B(x,t)}\frac{|f(y)|}{|y-x|^{\frac{1}{2}}}dy+\frac{1}{2\pi t^{\frac{1}{2}}}\int_{B(x,t)}\frac{|g(y)|}{|y-x|^{\frac{1}{2}}}dy+\frac{1}{4\pi t^{\frac{3}{2}}}\int_{B(x,t)}|\bigtriangledown f(y)|^2dy+\frac{1}{4\pi t^{\frac{3}{2}}}\int_{B(x,t)}|y-x|dy.$$

e. [UTK PDE Prelim August 2016 Problem 6] Now suppose that $n = 3$. Prove that for any $0 < t < 1, x \in \mathbb{R}^3$,

$$|u(x,t)|\leq |f(x)|+\sup_{y\in B(0,1)}|g(y)|.$$

f. [UTK PDE Prelim August 2016 Problem 6] Now suppose that $n = 3$. Prove that for any $t \geq 1, x \in \mathbb{R}^3$,

$$|u(x,t)|\leq |f(x)|+\frac{3}{4\pi t^2}\int_{B(x,t)}|g(y)|dy + \frac{1}{4\pi t}\int_{B(x,t)}|\bigtriangledown g(y)|dy.$$

13.3. **Problem 3.** Let $n \geq 2, \Omega \subset \mathbb{R}^n$ be a C^∞ bounded domain, $T > 0$ be fixed, h be a vector $h := (h_1, \ldots, h_n)$ such that $h_j \in C(\bar{\Omega} \times [0,T])$. Let $f, g : \bar{\Omega} \to \mathbb{R}$ be $C^2(\Omega) \cap C^0(\bar{\Omega})$ functions and suppose $u \in C^2(\Omega \times (0,T))$ solves

$$\begin{cases} u_{tt} = \triangle u + \triangledown u \cdot h, x \in \Omega, 0 < t < T \\ u = 0, x \in \partial\Omega, 0 \leq t \leq t \\ u(x,0) = f(x), x \in \Omega \\ u_t(x,0) = g(x), x \in \Omega \end{cases}.$$

Define the energy

$$E(t) := \frac{1}{2}\int_\Omega |\triangledown u|^2 + u_t^2 dx$$

and define $M := \sup_{\Omega \times [0,T]} |h|$ (notice this quantity is finite).

a. Prove that

$$E'(t) \leq \frac{M}{2}\int_\Omega |\triangledown u|^2 + u_t^2 dx + \frac{1}{2}\int_{\partial\Omega} \left|\frac{\partial u}{\partial n}\right|^2 + |u_t|^2 dS.$$

b. Prove that

$$E(t) \leq \frac{M}{2}\int_{\Omega\times[0,T]} |\triangledown u|^2 + u_t^2 dxdt + \frac{1}{2}\int_{\partial\Omega\times[0,T]} \left|\frac{\partial u}{\partial n}\right|^2 + |u_t|^2 dSdt + \frac{1}{2}\int_\Omega |\triangledown f|^2 + g^2 dx.$$

13.4. **Problem 4.** Let $g \in L^p(\mathbb{R}^3 \times (0,\infty))$ for some $1 < p < \infty$ and let $u \in C^2(\mathbb{R}^3 \times (0,\infty)) \cap C^0(\mathbb{R}^3 \times [0,\infty))$ solve

$$\begin{cases} u_{tt} - \triangle u = g(x,t), x \in \mathbb{R}^3, t > 0 \\ u(x,0) = 0, x \in \mathbb{R}^3 \\ u_t(x,0) = 0, x \in \mathbb{R}^3 \end{cases}$$

a. Show that for any $x \in \mathbb{R}^3, t > 0$, we have the estimate

$$|u(x,t)| \leq \frac{||g||_{L^p(\mathbb{R}^3\times(0,\infty))} t^{2-\frac{2}{p}}}{(4\pi)^{\frac{1}{p}}\left(2 - \frac{2}{p}\right)}.$$

b. If you replicate the same procedure in part a for $p = 1$, what happens?

13.5. **Problem 5.** a. Show that

$$u(x,t) = \frac{1}{2\pi}\int_0^t \int_{B(x,t-s)} \frac{e^s}{((t-s)^2 - |y-x|^2)^{\frac{1}{2}}} dyds$$

solves the PDE

$$\begin{cases} u_{tt}(x,t) - \triangle u(x,t) = e^t, (x,t) \in \mathbb{R}^2 \times \{t=0\} \\ u(x,0) = 0, \qquad\qquad\qquad (x,t) \in \mathbb{R}^2 \times \{t=0\} \\ u_t(x,0) = 0, \qquad\qquad\qquad (x,t) \in \mathbb{R}^2 \times \{t=0\} \end{cases}$$

b. Show that

$$u(x,t) = \frac{1}{2\pi}\int_0^t \int_{B(x,t-s)} \frac{e^s}{((t-s)^2 - |y-x|^2)^{\frac{1}{2}}} dyds + \frac{1}{2\pi}\int_{B(x,t)} \frac{|y|^2}{(t^2 - |y-x|^2)^{\frac{1}{2}}} dy$$

solves the PDE

$$\begin{cases} u_{tt}(x,t) - \triangle u(x,t) = e^t, (x,t) \in \mathbb{R}^2 \times \{t=0\} \\ u(x,0) = 0, \qquad\qquad\qquad (x,t) \in \mathbb{R}^2 \times \{t=0\} \\ u_t(x,0) = |x|^2, \qquad\qquad (x,t) \in \mathbb{R}^2 \times \{t=0\} \end{cases}$$

13.6. **Problem 6.** Let u be a C^2 solution to

$$u_{tt} - \triangle u = u^2$$

on $\mathbb{R}^n \times (0,\infty)$. We define, for a fixed $(x_0,t_0) \in \mathbb{R}^n \times (0,\infty)$, the energy

$$E(t) := \frac{1}{2}\int_{B(x_0,t_0-t)} u_t^2 + |\triangledown u|^2 dx$$

Show that for any $0 < t < t_0$,

$$E'(t) \leq \frac{1}{2}\int_{B(x_0,t_0-t)} u_t^2 + u^4 dx$$

13.7. **Problem 7.** Suppose u is a C^2 solution to

$$u_{tt} - \triangle u = 0$$

on $\mathbb{R}^3 \times (0,\infty)$ where $u(x,0) = f(x), u_t(x,0) = g(x)$ are continuous initial data on $\mathbb{R}^3$, where f is also of class C^1 on $\mathbb{R}^3$.

a. Prove for $0 < t < 1$,

$$u(0,t)^2 \leq \frac{3}{16\pi^2 t^4}\int_{\partial B(0,t)} f(y)^2 + g(y)^2 + |\triangledown f(y)|^2 dS(y)$$

b. Prove for $t \geq 1$,

$$u(0,t)^2 \leq \frac{1}{16\pi^2 t^2}\int_{\partial B(0,t)} (f(y) + g(y) + \triangledown f(y) \cdot y)^2 dS(y)$$

13.8. **Problem 8.** Let $a, R > 0$, consider a solution $u \in C^1((0,R) \times (0,\infty))$ to

$$\begin{cases} u_t + au_x = f(x,t), 0 < x < R, t > 0 \\ u(0,t) = 0, & t > 0 \\ u(x,0) = 0, & 0 < x < R \end{cases}$$

a. [UTK PDE Prelim Fall 2019 Problem 2] Prove for $t > 0$,

$$\int_0^R u^2(x,t)dx \leq e^t \int_0^t \int_0^R f^2(x,s)dxds$$

b. [My variant] Also prove for $t > 0$,

$$\frac{d}{dt}\int_0^R u^4(x,t)dx \leq 2\int_0^R f(x,t)^2 + u(x,t)^6 dx$$

13.9. **Problem 9.** Let $a, R > 0$, consider a solution $u \in C^1((0,R) \times (0,\infty))$ to

$$\begin{cases} u_{tt} + au_{xx} = f(x,t), & 0 < x < R, t > 0 \\ u(0,t) = u(R,t) = 0, & t > 0 \\ u(x,0) = 0, & 0 < x < R \end{cases}$$

Prove that for any $t > 0$,

$$\frac{d}{dt}\int_0^R uu_t dx \leq \frac{1}{2}\int_0^R u^2(x,t) + f^2(x,t)dx$$

13.10. **Problem 10.** Consider the initial-value problem for this nonlinear wave equation:

$$\begin{cases} u_{tt} - \triangle u + u^5 = 0, (x,t) \in \mathbb{R}^n \times (0,\infty) \\ u(x,0) = a_0(x), & x \in \mathbb{R}^n \\ u_t(x,0) = a_1(x) & x \in \mathbb{R}^n \end{cases}$$

Arbitrary scalings for solutions to hyperbolic PDEs come in the form

$$u_\lambda(x,t) = \lambda^\beta u(\lambda x, \lambda t)$$

for $\lambda > 0$. There exists a value of β for which u solving the PDE implies $u_\lambda(x,t)$ solves the same PDE (with scaled initial values) for all $\lambda > 0$. Find the value of β.

13.11. **Problem 11.** Suppose $u_0 \in C_C^\infty(\mathbb{R}^n)$ and that u solves the Cauchy problem for the wave equation,

$$\begin{cases} u_{tt} - \triangle u = 0, & (x,t) \in \mathbb{R}^n \times (0,\infty) \\ u(x,0) = u_0(x), & x \in \mathbb{R}^n \end{cases}$$

Let $p, q \in (1,\infty)$ and consider the inequality

$$||u(\cdot,t)||_{L^p(\mathbb{R}^n)} \leq \frac{N}{t^\alpha}||u_0||_{L^q(\mathbb{R}^n)}$$

for $t > 0$. Let N, α both be positive and depending only on n, p, and q. Suppose N is known and any solution to the given PDE satisfies this inequality with the corresponding initial data. Find α in terms of n, p, and q.

13.12. **Problem 12.** [This is a generalization of UTK PDE Prelim August 2019 Problem 8] Let $u \in C^2(\mathbb{R}^n \times [0,\infty))$ solve

$$u_{tt} = \triangle u$$

in $\mathbb{R}^n \times (0,\infty)$. Suppose $r(t)$ is a $C^1((0,\infty))$ function that is always positive. Define the energy

$$E(t) := \frac{1}{2}\int_{B(0,r(t))} |u_t(x,t)|^2 + |\triangledown u(x,t)|^2 dx$$

a. Prove that

$$E'(t) = \int_{B(0,r(t)} u_t(x,t)u_{tt}(x,t) + \sum_{i=1}^n u_{x_i}(x,t)u_{x_i t}(x,t)dx + \frac{1}{2}r'(t) + \int_{\partial B(0,r(t))} u_t^2(x,t) + |\triangledown u(x,t)|^2 dS(x).$$

b. Suppose that $r'(t) + 1 \leq 0$ for all t. Prove that $E'(t) \leq 0$ for all $t > 0$.

13.13. **Problem 13.** a. [Henry Simpson] Suppose $u \in C^2(\mathbb{R}^3 \times (0,\infty))$ solves

$$\begin{cases} u_{tt} - \triangle u = 0, (x,t) \in \mathbb{R}^3 \times (0,\infty) \\ u(x,0) = f(x), x \in \mathbb{R}^3 u_t(x,0) = g(x), x \in \mathbb{R}^3 \end{cases}$$

where $f, g \in C^0(\mathbb{R}^3)$. Define

$$M_f(x,t) = \fint_{\partial B(x,t)} f(y)dS(y)$$
$$M_g(x,t) = \fint_{\partial B(x,t)} g(y)dS(y)$$

and show that $u(x,t) = \frac{\partial}{\partial t}(tM_f(x,t)) + tM_g(x,t)$ when $x \in \mathbb{R}^3, t > 0$.

b. [My extension] Use part a to obtain an analogous formula for $u_t(x,t)$ when $x \in \mathbb{R}^3, t > 0$.

13.14. **Problem 14.** Let $f \in C^1(\mathbb{R})$ where $f' \in L^1(\mathbb{R})$ and $f(0) = 0$. Suppose $\phi, \psi \in C^2(\bar{\Omega})$ and $u \in C^2(\bar{\Omega_T})$ solves

$$\begin{cases} u_{tt} - \triangle u = f(u), (x,t) \in \overline{\Omega_T} \\ u(x,0) = \phi(x), \quad x \in \Omega \\ u_t(x,0) = \psi(x), \quad x \in \Omega \\ u = 1, \partial\Omega \times (0,T] \end{cases}$$

Define the energy

$$E(t) := \int_\Omega u_t^2 + |\bigtriangledown u|^2 + u^2 dx$$

Show that

$$E'(t) \leq \frac{||f'||_{L^1(\mathbb{R})}+1}{2} E(t)$$

for each t.

13.15. **Problem 15.** Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $T > 0$, $\Omega_T := \Omega \times (0,T)$, and let u solve the PDE

$$\begin{cases} u_{tt} - \triangle u + u^5 = f(x,t), (x,t) \in \Omega_T \\ u(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,T) \\ u_t(x,t) = \psi, u(x,0) = \phi, (x,t) \in \Omega \times \{t=0\} \end{cases}$$

where $u, f \in C^\infty(\Omega_T)$, $\psi, \phi \in C^\infty(\Omega)$. Define the energy

$$E(t) := \int_\Omega \frac{1}{2} u_t^2 + \frac{1}{2}|\bigtriangledown u|^2 + \frac{1}{6} u^6 dx$$

a. Prove that

$$E(t) = E(0) + \int_{\Omega_T} f u_t dx$$

b. Prove the continuous dependence inequality

$$\frac{1}{4}||u_t||^2_{L^2(\Omega_T)} + \frac{1}{2}||\bigtriangledown u||^2_{L^2(\Omega_T)} + \frac{1}{6}||u||^6_{L^6(\Omega_T)} \leq \frac{T}{2}||\psi||^2_{L^2(\Omega_T)} + \frac{T}{2}||\bigtriangledown\phi||^2_{L^2(\Omega_T)} + \frac{T}{6}||\phi||^6_{L^6(\Omega_T)} + T^2||f||^2_{L^2(\Omega_T)}$$

c. Suppose $|f| \leq 1$ on Ω_T. Prove that for $0 < t < T$,

$$E'(t) \leq \left(\frac{3}{2} \int_0^t ||f||_{L^1(\Omega)} ||u_t||_{L^2(\Omega)} (||f_t||_{L^2(\Omega)} + ||u_t||_{L^2(\Omega)} + ||f||_{L^2(\Omega)} + ||u_{tt}||_{L^2(\Omega)} dt \right)^{\frac{1}{3}}$$

13.16. **Problem 16.** Let $u(x,t)$ be a C^2 compactly supported solution to

$$\begin{cases} u_{tt} - \triangle u = 0 \\ u(x,0) = 0 \\ u_t(x,0) = g(x) \end{cases}$$

on $\mathbb{R}^3 \times (0,\infty)$. Suppose $\int_{\mathbb{R}^3} g^2 dx < \infty$

a. [UTK PDE Prelim January 2015 Problem 6] Prove that

$$\int_0^\infty u(0,t)^2 dx \leq \frac{1}{4\pi} \int_{\mathbb{R}^3} g(x)^2 dx$$

b. [My variant] Prove this alternative bound for the same quantity:

$$\int_0^\infty u(0,t)^2 dx \leq \int_0^\infty \int_0^t \int_{\partial B(0,s)} \frac{2}{\pi s} g^2(y) + \frac{1}{8\pi^2 s} y \cdot \triangledown g(y) dS(y)$$

13.17. **Problem 17.** Extrapolate the statement and proof of the finite propagation speed for the wave equation in $\mathbb{R}^n \times (0,\infty)$ on page 84 of [Ev] to conclude a similar result for the PDE

$$u_{tt} = \triangle u - u^7$$

13.18. **Problem 18.** State the D'Alembert Formula for solution to wave equation in one dimension. Give an explicit example of initial data to indicate that in $n = 1$ the solution does not adhere to the time decay behavior we see in higher dimensions.

13.19. **Problem 19.** Consider the following BVP for the wave equation. Let $u \in C^2(\mathbb{R}^3 \times (0,\infty)) \cap C^0(\mathbb{R}^3 \times [0,\infty))$ solve

$$\begin{cases} u_{tt} = \triangle u, (x,t) \in \mathbb{R}^3 \times \{t = 0\} \\ u(x,0) = f(x), x \in \mathbb{R}^3 \\ u_t(x,0) = g(x), x \in \mathbb{R}^3 \end{cases}$$

and suppose that $f, g : \mathbb{R}^n \to \mathbb{R}$ are smooth. Prove that

$$u(x,t) = \frac{1}{4\pi} \int_{\partial B(0,1)} f(x+wt) + tg(x+wt) + tw \cdot \triangledown f(x+wt) dS(w)$$

and

$$u_t(x,t) = \frac{1}{4\pi} \int_{\partial B(0,1)} w \cdot \triangledown f(x+wt) + g(x+wt) + tw \cdot \triangledown g(x+wt) + w \cdot \triangledown f(x+wt) + tw \cdot \triangledown f_t(x+wt) dS(w)$$

for $x \in \mathbb{R}^3, t > 0$.

13.20. **Problem 20.** Suppose $u \in C^\infty(R^n)$ solves

$$u_{tt} - \triangle u + u^5 = 0$$

for $t \geq 0$, and define the energy

$$E(t) := \int_{\mathbb{R}^n} \frac{1}{2} u_t^2 + \frac{1}{2} |\triangledown u|^2 + \frac{1}{6} u^6 dx$$

a. Prove that the energy $E(t)$ is constant in time.
b. Prove that for any $t > 0$,

$$\int_{\mathbb{R}^n} |\triangledown u| + |u_t| + |u|^3 dx \leq \sqrt{6\alpha(n) E(0)} (R+t)^{\frac{n}{2}}$$

13.21. **Problem 21.** Suppose $u \in C^2(\mathbb{R} \times [0, \infty))$ satisfies

$$\begin{cases} u_{tt} = \triangle u, (x,t) \in \mathbb{R} \times \{t = 0\} \\ u(x,0) = 0, x \in \mathbb{R} \\ u_t(x,0) = f(x), x \in \mathbb{R} \end{cases}$$

where $f \in C^\infty(\mathbb{R})$ is compactly supported and $f' \in L^1(\mathbb{R})$. Show that for each $x \in \mathbb{R}, t > 0$,

$$|u(x,t)| \leq x||f'||_{L^1(\mathbb{R})}$$

13.22. **Problem 22.** The wave equation is traditionally written in the form

$$u_{tt} = c^2 \triangle u$$

where c is a positive constant.

a. Explicitly state what scaling property the wave equation has.

b. Explain why this scaling property essentially allows us to assume without loss of generality that $c = 1$.

14. Integrals (Complex-Valued)

14.1. **Problem 1.** Using the Residue Theorem, calculate each integral for $n \in \mathbb{N}^+$:

a. [UTK Analysis Prelim January 2013 Problem 6] $\int_0^{2\pi} \cos^{2n}(\theta) \frac{d\theta}{2\pi}$

b. $\int_0^{2\pi} \sin^{2n}(\theta) \frac{d\theta}{2\pi}$

NOTE: there are inductive techniques that also solve these without using complex variables; you may wish to check your work using these.

14.2. **Problem 2.** Let $f : \mathbb{C} \to \mathbb{C}$ be entire and let G_n be an increasing sequence of convex sets. Prove that for any contour Γ in $\cup_{n=1}^\infty G_n$, we have $\int_\Gamma f(z)dz = 0$.

14.3. **Problem 3.** a. Prove that if γ is a closed curve with $0, 1 \notin \text{int}(\gamma)$ then

$$\int_\gamma \frac{z^2 + z + 1}{z^2(z-1)^2} dz = 0$$

b. Prove that for any $k \in \mathbb{N}^+$ there exists a closed curve γ with $0 \in \text{int}(\gamma)$ and $1 \notin \text{int}(\gamma)$ such that

$$\int_\gamma \frac{z^2 + z + 1}{z^2(z-1)^2} dz = 2\pi i k$$

c. Prove that for any $k \in \mathbb{N}^+$ there exists a closed curve γ with $0, 1 \in \text{int}(\gamma)$ such that

$$\int_\gamma \frac{z^2 + z + 1}{z^2(z-1)^2} dz = 2\pi i k$$

14.4. **Problem 4.** If C denotes the unit circle oriented counterclockwise, calculate

$$\frac{1}{2\pi i}\int_C \frac{z^5}{2z^6-1}dz$$

14.5. **Problem 5.** [UTK Analysis Prelim Fall 2006 Problem 1] Calculate

$$\int_{|z|=1} e^{\frac{1}{z}}dz$$

Note: as the original problem statement did not specify, assume the integral is with a counterclockwise orientation around the boundary of the unit disc.

14.6. **Problem 6.** Describe under which set(s) of conditions Cauchy's Theorem is a direct corollary of the Argument Principle.

15. Integrals and Limit Theorems (Lebesgue)

15.1. **Problem 1.** a. State the following theorems: Monotone Convergence Theorem, Fatou's Lemma, Dominated Convergence Theorem.

b. Use the Monotone Convergence Theorem to prove Fatou's Lemma.

c. Use Fatou's Lemma to prove Dominated Convergence Theorem.

d. Use the Dominated Convergence Theorem to prove the Monotone Convergence Theorem without Fatou's Lemma.

e. Use Fatou's Lemma to prove the Monotone Convergence Theorem without the Dominated Convergence Theorem.

f. Use the Dominated Convergence Theorem to prove Fatou's Lemma without the Monotone Convergence Theorem.

15.2. **Problem 2.** Suppose $f_n : X \to \mathbb{R}$ are negative, measurable, decreasing functions and $f_n \to f$ as $n \to \infty$ pointwise. Show that

$$\lim_{n\to\infty}\int_X f_n^2 d\mu = \int_X f^2 d\mu.$$

15.3. **Problem 3.** Suppose $f_n, g_n, f, g : X \to \mathbb{R}$ are measurable with $f_n \to f$ and $g_n \to g$ uniformly on X and $|f_n|\leq |g_n|$ for all $n \in \mathbb{N}^+$ and $\lim_{n\to\infty}\int_X g_n d\mu = \int_X g d\mu$.

a. Prove that, using the Dominated Convergence Theorem,

$$\lim_{n\to\infty}\int_X f_n + g_n d\mu = \int_X f + g d\mu.$$

b. Using a different technique, prove the same claim but replacing the uniform convergence with a.e. pointwise convergence.

15.4. **Problem 4.** a. This problem appears in many textbooks and is worth working through at least once. It is called the **Riemann-Lebesgue Lemma**[1]. Prove that

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f(x)\cos(nx)dx = 0$$

whenever $f \in L^1(\mathbb{R})$.
b. Now suppose $f \in L^2(\mathbb{R})$ and show

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} f(x)\sqrt{|\cos(nx)|}dx = 0.$$

c. Now suppose $f' \in L^1(\mathbb{R})$ and show

$$\lim_{n\to\infty} n\int_{-\infty}^{\infty} f(x)\sin(nx)dx = 0.$$

15.5. **Problem 5.** a. Let $f_n : X \to \mathbb{R}$ be measurable and nonnegative. Then show that

$$\lim_{n\to\infty}\int \sup_{k\leq n} f_k d\mu = \int \text{limsup}_{n\to\infty} f_n d\mu.$$

b. Now suppose $f_n : X \to \mathbb{R}$ is just measurable. Now show that

$$\lim_{n\to\infty}\int \sup_{k\leq n} f_k^2 d\mu = \int \text{limsup}_{n\to\infty} f_n^2 d\mu.$$

c. Finally let $f_n : [0,1] \to \mathbb{R}$ be measurable. Show that

$$\int_0^1 \text{limsup}_{n\to\infty} f_n^2 d\mu \geq \left(\lim_{n\to\infty}\int_0^1 \sup_{k\leq n} f_k d\mu\right)^2.$$

15.6. **Problem 6.** Let $f_n : K \to \mathbb{R}$ where K is a compact subset of $\mathbb{R}^n$. Suppose $f_n \to 0$ uniformly on K as $n \to \infty$. Show that

$$\lim_{n\to\infty}\int_K f_n d\mu = 0.$$

15.7. **Problem 7.** Evaluate the following limit-sum expressions:
a.

$$\lim_{n\to\infty}\sum_{m=1}^{\infty}\frac{n^2 m}{n^2 m^3 + 1}$$

b.

$$\lim_{n\to\infty}\sum_{m=1}^{\infty}\frac{\ln(n)^4 m}{nm^3 + 1}$$

[1] there are several statements under this name but they claim very similar things

c.

$$\lim_{n\to\infty}\sum_{m=1}^{\infty}\frac{n^{m+1}}{(n+2)((2n)^m+2)}$$

15.8. **Problem 8.** This one is a variant of Problem 1 from the UTK Analysis Prelim January 2015. Let $f_n(x) := \frac{e^{\sin\left(\frac{x^2}{n^2}\right)}}{1+x}$ and calculate the following limit for any parameter $\alpha > 1$:

$$\lim_{n\to\infty}\int_0^n f_n^\alpha dm.$$

15.9. **Problem 9.** The following is UTK Analysis Prelim January 2013 Problem 7.

Let f_n, f be positive, integrable functions on the measure space $(X, \mathcal{M}, \mu)$. Assume that $f_n \to f$ pointwise and $\int_X f_n d\mu \to \int_X f d\mu < \infty$. Show that $\int_E f_n d\mu \to \int_E f d\mu$ for all $E \in \mathcal{M}$

Even though this problem isn't my own I want to highlight two separate ways to prove it. First, there is the way which I think the problem was intended to be solved.

a. State and prove the Generalized Dominated Convergence Theorem.

b. Use that theorem to solve this problem.

Now I present a method which I believe works but does not make use of the $\int_X f_n d\mu \to \int_X f d\mu$ assumption.

c. Let $E \in \mathcal{M}$ be fixed and let $g_n := \chi_E f_n$ for each $n \in \mathbb{N}^+$. Why must $|g_n| \leq 2|f|$ on X for all n larger than some positive integer K?

d. Define $h := 2f + \sum_{i=1}^K g_i$. Show that $h \in L^1(\mu)$.

e. Use this function h and the regular Dominated Convergence Theorem to solve the problem.

15.10. **Problem 10.** Let $h : \mathbb{R} \to \mathbb{R}$ be a bounded function in $L^1(\mathbb{R})$ where $\lim_{|x|\to 0} h(x) = 0$. Prove that

$$\lim_{n\to\infty} h(nx) = 0$$

15.11. **Problem 11.** Let $g : [0,1] \to \mathbb{R}$ be measurable and in $L^1(m)$.

a. [Remus Nicoara] If $\int_{(a,b)} gdm = b - a$ for any $0 < a < b < 1$, prove that $g(x) = 1$ a.e. on $[0,1]$.

b. [Remus Nicoara] If $\int_{(a,b)} gdm \geq b - a$ for any $0 < a < b < 1$, prove that $g(x) \geq 1$ a.e. on $[0,1]$.

15.12. **Problem 12.** [Remus Nicoara] Calculate, with proof, the limit $\lim_{N\to\infty}\int_0^N \frac{N+\cos^2 x}{N^2+2N\cos x^2}dx$.

15.13. **Problem 13.** [Remus Nicoara] Let μ be a Borel measure on $\mathbb{R}$ such that $\mu(\mathbb{R}) = 1$ and

$$\lim_{T\to 0}\frac{1-\cos(Tx)}{T^2}d\mu(x) = 0$$

Show that $\mu(\{0\}) = 1$.

15.14. **Problem 14.** [Remus Nicoara] Suppose $\{f_n\}_{n=1}^{\infty}, f : X \to \mathbb{R}$ are measurable with $\mu(X) < \infty$. Show the following are equivalent;

i) $f_n \to f$ in measure.

ii) $\int_X \frac{|f_n - f|}{1+|f_n - f|} d\mu \to 0$ as $n \to \infty$.

15.15. **Problem 15.** [Remus Nicoara] Suppose $f_n \to f$ in measure and $\{f_n\}_{n=1}^{\infty}, f$ are all non-negative and measurable. Prove $\int f d\mu \leq \text{liminf}_{n\to\infty}$.

15.16. **Problem 16.** [Remus Nicoara] Let $f \in L^p([0,\infty])$ for some $1 \leq p \leq \infty$. Then calculate

$$\lim_{N\to\infty} \int_0^{\infty} e^{-Nx^2} f(x)dx$$

15.17. **Problem 17.** a. [Remus Nicoara] Let μ, ν be positive and finite on $(X, \mathcal{M})$. Show there exists a function f that is $\mathcal{M}$-measurable such that for any $E \in \mathcal{M}$,

$$\int_E (1-f)d\mu = \int_E f d\nu$$

and show $0 \leq f \leq 1$ a.e. with respect to μ.

b. [My variant] Can this result be extended to the case where μ and ν are σ-finite?

15.18. **Problem 18.** [Remus Nicoara] For any E Lebesgue measurable, let $\mu(E) := \int_E \frac{1}{1+x^2} dm(x)$.

a. Show that $m << \mu$.

b. Find $\frac{dm}{d\mu}$.

15.19. **Problem 19.** [Remus Nicoara] Let $X = [0,1]$. Let c denote the counting measure on X. Let $\mathcal{M}$ be the σ-algebra of Lebesgue measurable subsets of X. Show that $m << c$ but there is no f measurable with respect to $\mathcal{M}$ such that $m(E) = \int_E f dc$ for all $E \in \mathcal{M}$. Explain why this does not contradict the Radon-Nikodym Theorem.

15.20. **Problem 20.** [Remus Nicoara] Let m be Lebesgue measure on $(0,\infty)$. Let $\mathcal{M}$ be the σ-algebra generated on X by intervals of the form $\left(\frac{1}{N+1}, \frac{1}{N}\right]$ for $N \in \mathbb{N}^+$. Let $\lambda(E) := \int_E 2x^2 dm(x)$ for all $E \in \mathcal{M}$. Does there exist a $\mathcal{M}$-measurable function f on X such that $\lambda(E) = \int_E f dm$ for all $E \in \mathcal{M}$?

15.21. **Problem 21.** Suppose μ is a positive measure on a sigma-algebra $\mathcal{A} \subset 2^{\mathbb{R}}$ such that

$$\mu(E) = \int_E \sin(x)dm(x)$$

for each $E \in \mathcal{A}$.

If μ is taken to be σ-finite, what restriction(s) must hold for $\mathcal{A}$?

16. Interpolation Inequalities

16.1. **Problem 1.** a. Find a relationship between $p, q \in (1, \infty)$ such that the scaling $u_\lambda(x) := u(\lambda x), v_\lambda(x) = v(\lambda x)$ preserves the inequality

$$||uv||_{L^1(\mathbb{R}^n)} \leq ||u||_{L^p(\mathbb{R}^n)} ||v||_{L^q(\mathbb{R}^n)}$$

where $u \in L^p(\mathbb{R}^n), v \in L^q(\mathbb{R}^n)$.

b. Find a relationship between $p, q \in (1, \infty)$ such that the scaling $u_\lambda(x) := u(\lambda x), v_\lambda(x) = v(\lambda x)$ preserves the inequality

$$||uv||_{L^1(\mathbb{R}^n)} \leq ||\bigtriangledown u||_{L^p(\mathbb{R}^n)} ||v||_{L^q(\mathbb{R}^n)}$$

where $u \in L^p(\mathbb{R}^n) \cap C_C^\infty(\mathbb{R}^n), v \in L^q(\mathbb{R}^n)$.

16.2. **Problem 2.** Let $p, q, r \in (0, 1)$. Find a relationship between p, q, r such that the scaling $f_\lambda(x) := f(\lambda x), g_\lambda(x) := g(\lambda x)$ preserves the inequality

$$||f * g||_{L^r(\mathbb{R})} \geq ||f||_{L^p(\mathbb{R})} ||g||_{L^q(\mathbb{R})}$$

Throughout assume all quantities in question are finite.

16.3. **Problem 3.** a. Prove that

$$\int_0^1 \sum_{n=1}^k e^{\frac{x^2}{n}} \left(2 - \frac{x}{n^2}\right) dx \leq \sum_{n=1}^k \int_0^1 e^{\frac{2x^2}{n}} dx \int_0^1 \left(2 - \frac{x}{n^2}\right)^2 dx$$

for any $k \in \mathbb{N}^+$.

b. Why is this result meaningless if we send $k \to \infty$?

16.4. **Problem 4.** [Remus Nicoara] Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f \in L^p(X) \cap L^q(X)$ for $1 \leq p < q < \infty$. Show that $f \in L^r(X)$ for any $p \leq r \leq q$.

16.5. **Problem 5.** Let $\Omega \subset \mathbb{R}^n$ be an open, bounded domain and let u be a harmonic function on Ω. Then prove there exists constant C depending on n and Ω such that

$$||u||_{L^\infty(\Omega)} \leq C ||\bigtriangledown u||_{L^2(\Omega)}$$

16.6. **Problem 6.** a. [UTK Analysis Prelim Fall 2018 Problem 4] Let $(\mathbb{R}, \mathcal{M}, m)$ be the Lebesgue measure space. If $f \in L^1(m)$, $g \in L^p(m)$ for a $p \in [1, \infty)$, prove that

$$||f * g||_{L^p(\mathbb{R})} \leq ||f||_{L^1(\mathbb{R})} ||g||_{L^p(\mathbb{R})}$$

b. [My variant] Show the claim of part a is still true if $p = \infty$.

16.7. **Problem 7.** [Found on MathStackExchange] Suppose $(X, \mathcal{M}, \sigma)$ is a finite measure space and $f : X \to \mathbb{R}$ is measurable with $f \in L^\infty(X) \cap L^q(X)$ for some $q > 1$. Show that

$$\lim_{p \to \infty} ||f||_{L^p(X)} = ||f||_{L^\infty(X)}$$

17. Laplace's Equation and Harmonic Functions

17.1. **Problem 1.** a. Let u be harmonic in $\mathbb{R}^n$ with $u \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$ for some $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Prove that u is identically zero.

b. Let u be harmonic in $\mathbb{R}^n$ with $u \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$. Prove that u is identically zero.

c. [Han p. 144 Problem 4.5] Let u be harmonic in $\mathbb{R}^n$ with $u \in L^p(\mathbb{R}^n)$. Prove that u is identically zero.

Note: Parts a and b are weaker versions of c, so a proof for c will work for a and b, but there are alternative proofs that use the additional assumptions.

17.2. **Problem 2.** a. Let $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ and let $\alpha \in C^0(\partial\Omega)$, where u solves

$$\begin{cases} \triangle u = 0, x \in \Omega \\ \frac{\partial u}{\partial n} + \alpha u = 0, x \in \partial\Omega \end{cases}.$$

Prove that $\int_\Omega \alpha u^2 \leq 0$.

b. If α is positive on a positive [Lebesgue] measure subset of Ω and zero everywhere else, deduce that the only solution to the PDE in part a is the zero solution.

c. Let $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ and let $\alpha \in C^0(\partial\Omega)$, where u solves

$$\begin{cases} \triangle u = 0, x \in \Omega \\ \frac{\partial u}{\partial n} + \alpha u^2 = 0, x \in \partial\Omega \end{cases}.$$

Again assume $\alpha \geq 0$ on $\partial\Omega$. Prove that

$$\int_\Omega e^{-u^2} u \left(\sum_{i=1}^n u_{x_i}^2 \right) dx \geq 0.$$

17.3. **Problem 3.** [UTK PDE Prelim August 2016 Problem 2] Suppose $n \geq 2, R > 0, B(0,R) \subset \mathbb{R}^n$, and $u : \overline{B(0,R)} \to \mathbb{R}$ satisfies $u \in C(\overline{B(0,R)})$, u harmonic on $B(0,R)$, and $u \geq 0$ on $B(0,R)$.

a.[2]Prove that, for all $x \in B(0,R)$,

$$\frac{(R-|x|)R^{n-2}}{(R+|x|^{n-1})} u(0) \leq u(x) \leq \frac{(R+|x|)R^{n-2}}{(R-|x|)^{n-1}} u(0)$$

b. Prove also, for $x \in B(0,R)$ and $j = 1, 2, \ldots, n$,

$$|u_{x_j}(x)| \leq \frac{(2n+2)R^{n-1}}{(R-|x|)^n} u(0)$$

c. [My extension] Suppose that u possesses all the same properties as in parts a and b, except that it may take negative values. Prove there exists $M_R > 0$ depending on R (and u) such that

---

[2]I have also seen this problem in other places, and it is called the explicit form of Harnack's Inequality. It was also a homework problem in a partial differential equations course I took during my last semester of undergraduate study.

$$\frac{(R-|x|)R^{n-2}}{(R+|x|^{n-1})}(u(0)+M_R)-M_R \leq u(x) \leq \frac{(R+|x|)R^{n-2}}{(R-|x|)^{n-1}}(u(0)+M_R)-M_R$$

d. [My extension] Suppose that $n \geq 3$ and that $u \in C(\overline{B(0,R)})$ and $u \geq 0$, but no longer assume u is harmonic. Prove that if

$$(R|x|+(n-2)(R+|x|))(R-|x|) \geq (R+|x|)(n-1)$$

then we obtain the same upper bound as in part a.

e. Use part a to prove Liouville's Theorem: a function that is harmonic and bounded in all of $\mathbb{R}^n$ is a constant function.

17.4. **Problem 4.** **NOTE:** This problem is attributed to Evans' PDE book in various locations; I have compiled a family of related problems into one place. They are collectively referred to as the *mean value property*.

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set, let $B(x,r)$ be an open ball in Ω. Let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$.

a. Show that if u is harmonic, then $u(x) = ⨍_{\partial B(x,r)} u(y)dS(y)$.

b. Show that if u is subharmonic, then $u(x) \leq ⨍_{\partial B(x,r)} u(y)dS(y)$.

c. Show that if u is superharmonic, then $u(x) \geq ⨍_{\partial B(x,r)} u(y)dS(y)$.

17.5. **Problem 5.** a. Carefully state the mean-value property for harmonic functions.

b. Prove the boundary integral version of the mean-value property using Poisson's Integral Formula.

NOTE: This is not the standard proof covered in texts such as [Ev].

17.6. **Problem 6.** Let $\Omega \subset \mathbb{R}^n$ be open and bounded, let $\{u_m\}_{m=1}^{\infty} : \Omega \to \mathbb{R}$ be harmonic in Ω.

a. If the functions $\{u_m\}_{m=1}^{\infty}$ are all nonnegative and the partial sums have a pointwise limit $u(x) = \sum_{m=1}^{\infty} u_m(x)$, prove u is harmonic on Ω.

b. If the functions $\{u_m\}_{m=1}^{\infty}$ are such that the sum $u(x) = \sum_{m=1}^{\infty} u_m(x)$ is absolutely convergent, prove u is harmonic on Ω.

17.7. **Problem 7.** a. [UTK PDE Prelim Fall 2013 Problem 4] Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Suppose $\{u_n\}_{n=1}^{\infty}$ is a sequence of harmonic functions on Ω such that

$$\int_{\Omega} |u_n(x) - u_m(x)|^2 dx \to 0$$

as $\max\{m,n\} \to \infty$. Prove that u_n converges (in any sense you like) to a harmonic function in Ω.

b. [Tuoc Phan] Show the above result continues to hold if we replace the 2 in the limiting behavior with an exponent $1 \leq p < \infty$.

17.8. **Problem 8.** [Purdue PDE Prelim Fall 2005 Problem 3] Let Φ be the fundamental solution of the Laplace Equation in $\mathbb{R}^n$, and let $f \in C_0^\infty(\mathbb{R}^n)$. Then the convolution

$$u(x) = (\Phi * f)(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y) dy$$

solves $-\triangle u = f$ in $\mathbb{R}^n$. Show that if f is radial and is supported in $B(0,R)$ then $u(x) = c\Phi(x)$ for all $x \in \mathbb{R}^n \setminus B(0,R)$, where $c = \int_{\mathbb{R}^n} f(y)dy$.

17.9. **Problem 9.** a. [Purdue PDE Prelim Fall 2014 Problem 1] Let $u \in C^2(\mathbb{R}^n)$ be harmonic in $\mathbb{R}^n$. Prove that u is constant if there exists a constant $C > 0$ such that

$$\int_{B(x,1)} |u(y)| dy \leq C$$

for any $x \in \mathbb{R}^n$.

b. [My variant] Let $u \in C^2(\mathbb{R}^n)$ be harmonic in $\mathbb{R}^n$. Prove that u is constant if there exists a constant $C > 0$ such that

$$\int_{\partial B(x,1)} |u(y)| dy \leq C$$

for any $x \in \mathbb{R}^n$.

c. [My variant] Let $u \in C(\mathbb{R}^n)$ be nonnegative and subharmonic in $\mathbb{R}^n$. Prove that u is bounded if there exists a constant $C > 0$ such that

$$\int_{\partial B(x,1)} |u(y)| dy \leq C$$

for any $x \in \mathbb{R}^n$.

17.10. **Problem 10.** Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set that is supported in $B(0,R)$. Let $G(x,y)$ be the Green's Function for the Laplacian operator, let Φ denote the fundamental solution to Laplace's Equation, and let $h(x,y)$ be the corrector function associated with the Green's Function G. Suppose $n \geq 3$. Prove the following estimates for any $x, y \in \Omega$ with $x \neq y$.

a. $\min_{y \in \partial\Omega} \frac{1}{n(n-2)\alpha(n)|x-y|^{n-2}} \leq h(x,y) \leq \max_{y\in\partial\Omega} \frac{1}{n(n-2)\alpha(n)|x-y|^{n-2}}$

b. $h(x,y) \geq \frac{1}{n(n-2)\alpha(n)(2R)^{n-2}}$

c. $h(x,y) \leq \frac{1}{n(n-2)\alpha(n)||x|-|y||}$.

17.11. **Problem 11.** Suppose $u \in C^2(\mathbb{R}^n)$ is harmonic in $\mathbb{R}^n$. Then prove for any $r \in (0, \infty)$,

$$\int_{\partial B(0,1)} w \cdot \triangledown u(rw) dS(w) = n \int_{B(0,1)} x \cdot \triangledown u(rx) dx$$

18. L^p SPACES AND CONVOLUTIONS

18.1. **Problem 1.** a. Suppose $f \in L^p(0,1)$ for a $1 \leq p \leq \infty$. Prove that $f \in L^1(0,1)$.
b. Let $1 < p < q < \infty$. Find a function in $L^p(\mathbb{R})$ that is not in $L^q(\mathbb{R})$.
c. Let $1 < p < q < \infty$. Find a function in $L^q(\mathbb{R})$ that is not in $L^p(\mathbb{R})$.
d. Let $1 < p < q < \infty$. Prove that $L^q((0,1)) \subset L^p((0,1))$.

18.2. **Problem 2.** a. State and prove Chebychev's Inequality.
b. Let $(\mu, \mathcal{A}, X)$ be a finite measure space (in fact for simplicity assume $\mu(X) = 1$). Prove that for a measurable function $f : X \to \mathbb{R}$,

$$\text{limsup}_{p\to\infty}\mu(\{x \in X : |f(x)| \geq 1\})^{\frac{1}{p}} \leq ||f||_{L^\infty(\mu)}$$

18.3. **Problem 3.** Suppose (X, A, μ) is a measure space where $f \in L^2(\mu)$ and $g \in L^4(\mu)$. Show then that $fg \in L^1(\mu)$ and

$$||fg||_{L^1(\mu)} \leq ||f||^3_{L^2(\mu)}||g||^4_{L^4(\mu)}$$

18.4. **Problem 4.** Suppose X is a Lebesgue measurable set where $m(X) = \infty$.
a. [Remus Nicoara] Show that for given $\alpha_1, \alpha_2, ... > 0$ there are disjoint sets $X_1, X_2, ... \subset X$ such that $m(X_i) = \alpha_i$.
b. [Remus Nicoara] Show there exists an f Lebesgue measurable on X such that $f \in L^p(X)$ for all $p \geq 1$ and $f \notin L^\infty(X)$.
c. [My variant] Fix a "critical" exponent $1 < p* < \infty$ and show that there exists an $f \in L^p(X)$ for all $1 \leq p \leq p*$ and $f \notin L^{p*}(X)$.

18.5. **Problem 5.** [Remus Nicoara] Let $(X, \mathcal{M}, \mu)$ be a measure space, $1 \leq p \leq \infty$, and let $\{f_n\}_{n=1}^\infty$ be a collection of measurable functions in $L^p(\mu)$ where $\sum_{n=1}^\infty ||f_n||_{L^p(\mu)} < \infty$ and $\sum_{n=1}^\infty |f_n| < \infty$. Prove that under these conditions, Minkowski's Inequality for finitely many of the $\{f_n\}_{n=1}^\infty$ functions can be extended to all of them.

18.6. **Problem 6.** [Remus Nicoara] Let $(X, \mathcal{M}, \mu)$ be a finite measure space and $M > 0$ be such that $\{f_N\}_{N=1}^\infty, f, g \in L^2(\mu)$ such that $f_n \to f$ almost everywhere and $||f_N||_{L^2(\mu)} \leq M \ \forall N \geq 1$. Show that

$$\int_X f_N g d\mu \to \int_X f g d\mu$$

for $N \to \infty$.

19. LOGARITHMS AND ROOTS

19.1. **Problem 1.** Let $\theta \in [0, 2\pi)$ and define $G_\theta := \mathbb{C} \setminus \{re^{i\theta}, r \geq 0\}$. Show there exists a branch of the [complex] logarithm and a branch of the argument function in G_θ. Why can we not include the origin in the set G_θ?

20. MEASURES

20.1. **Problem 1.** a. [UTK Analysis Prelim January 2015 Problem 4a] Let $\mathcal{M}$ denote the Lebesgue Measurable subsets of $[0,1]$. Let μ be a positive measure on $([0,1],\mathcal{M})$ such that $\mu(\{0\}) = 0$. Define

$$f(x) := \mu([0,x])$$

for any $0 \leq x \leq 1$. Assume f is absolutely continuous on $[0,1]$. Prove that $\mu << m$.
b. Using part a, extend the claim itself to the case where μ is a signed measure.

20.2. **Problem 2.** Using the construction of the Lebesgue-Stieltjes measure, show that the 1-dimensional Lebesgue measure of a countable subset of $\mathbb{R}$ is zero.

20.3. **Problem 3.** Consider a measure space $(X,\mathcal{A},\mu)$. We say that the space is σ-finite if:
i) There exist $E_1, E_2, \ldots \in \mathcal{A}$ with $\mu(E_i) < \infty$ and $\cup_{i=1}^{\infty} E_i = X$.
ii) There exist $F_1, F_2, \ldots \in \mathcal{A}$ with $\mu(F_i) < \infty$ and $F_i \nearrow X$.
Prove that these two definitions are equivalent.

20.4. **Problem 4.** Let μ, ν be finite measures on the measurable space $(X,\mathcal{A})$. Suppose a sequence of measurable functions $\{f_n\}$ converges to f in measure ν, and suppose $\mu << \nu$. Show that $\{f_n\}$ converges to f in measure μ as well.

20.5. **Problem 5.** Let $(X,\mathcal{A})$ be a measurable space and assume μ is a complex measure. That is, it is a map $\mu : \mathcal{A} \to \mathbb{C}$ that is bounded and possesses the following properties:
i) $\mu(A) = 0$
ii) For $A_1, A_2, \ldots \in \mathcal{A}$ disjoint, $\sum_{n=1}^{\infty} \mu(A_n) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right)$.
Now, suppose $\nu, \omega : \mathcal{A} \to \mathbb{R}$ are bounded functions such that

$$\mu(A) = \nu(A) + i\omega(A)$$

for each $A \in \mathcal{A}$.
a. Prove that ν and ω are uniquely determined by this property.
b. Prove that ν and ω are real-valued, finite measures.
c. Define a notion of σ-finite for complex-valued measures that is analogous to the one for real-valued measures. Then use said notion to generalize parts a and b.

20.6. **Problem 6.** Suppose $A_1, A_2, \ldots$ are Lebesgue measurable.
a. [Remus Nicoara] If $\sum_{n=1}^{\infty} m(A_n) < \infty$, prove that almost all x belong to at most finitely many of $A_1, A_2, \ldots$
b. [My extension] One can produce counterexamples to show that this no longer holds if $\sum_{n=1}^{\infty} m(A_n) = \infty$. Find an especially "good" counterexample where no x belong to at most finitely many of $A_1, A_2, \ldots$

20.7. **Problem 7.** a. Prove that the Lebesgue measure m on $\mathbb{R}$ is σ-finite.

b. Prove that for any E Lebesgue measurable with $m(E) = \infty$, there exist $E_1, E_2, ...$ such that $m(E_i) < \infty$ and $E_i \nearrow E$.

c. [Remus Nicoara] Let E be a Lebesgue measurable set with $m(E) = 1$. Prove for any $0 < t < 1$, there is an $F \subset E$ Lebesgue measurable with $m(F) = t$. Note: it is easy to see this works for any positive finite value of $m(E)$ with the same argument.

d. [My extension] Generalize part c to the case where $m(E) = \infty$. Prove your result.

20.8. **Problem 8.** For each part, either prove the claim is true or provide a counterexample.

a. Pointwise convergence implies convergence in measure. b. If f_n is a sequence of measurable functions converging in L^p to a function f on a set X, then for any two measurable subsets X_1, X_2 such that $X_1 \cup X_2 = X$, f_n will converge to f in L^p on both X_1 and X_2. c. Pointwise convergence on a finite measure space implies convergence in measure. d. Convergence in measure implies pointwise convergence. e. Pointwise convergence implies convergence in L^p for any $1 \leq p < \infty$. f. Pointwise convergence implies convergence in L^∞. g. Convergence in L^p implies convergence in measure.

20.9. **Problem 9.** [Remus Nicoara] Let X be a finite measure space, $f_N \to f$ in measure. Show then $f_N^2 \to f^2$ in measure also.

20.10. **Problem 10.** [Remus Nicoara] Let $f, g \geq 0$ be measurable on $[0, 1]$ with respect to Lebesgue Measure, where $\int_0^1 f dm = 2$, $\int_0^1 g dm = 1$, and $c > 4$ with $\int_0^1 f^2 dm \leq c$. Show that if $E = \{x \in [0, 1], f(x) > g(x)\}$, then $m(E) \geq \frac{1}{c}$.

20.11. **Problem 11.** [Remus Nicoara] Characterize the class of positive measures μ and ν where $\nu << \mu$ and $\nu \perp \mu$.

20.12. **Problem 12.** [Remus Nicoara] Prove the following properties of Radon-Nikodym Derivatives. Assume all measures are positive and σ-finite.

a. Let $\nu_1 << \mu$ and $\nu_2 << \mu$. Then

$$\frac{d(\nu_1 + \nu_2)}{d\mu} = \frac{d\nu_1}{d\mu} + \frac{d\nu_2}{d\mu}$$

b. Let $\eta << \nu << \mu$. Then

$$\frac{d\eta}{d\mu} = \frac{d\eta}{d\nu} \cdot \frac{d\nu}{d\mu}$$

c. Interpret part b in the special case where $\mu = \eta$.

20.13. **Problem 13.** [Remus Nicoara] Let $\mu, \nu \geq 0$ be finite measures on $(X, \mathcal{M})$ such that $\nu << \mu << \nu$. Let $f := \frac{d\nu}{d(\mu+\nu)}$. Show that $0 < f < 1$ a.e. with respect to μ.

20.14. **Problem 14.** [Remus Nicoara] Show the following properties hold where ν_1, ν_2, μ are positive measures.

a. If $\nu_1 << \mu$ and $\nu_2 \perp \mu$, then $\nu_1 \perp \nu_2$.

b. If $\nu_1 \perp \mu$ and $\nu_2 \perp \mu$ then $\nu_1 + \nu_2 \perp \mu$.

20.15. **Problem 15.** [Remus Nicoara] Show that if ν, μ, η are all positive measures with $\nu << \mu$, then

a. $\nu \times \eta << \mu \times \eta$

b. If also assuming $\nu \perp \mu$, then show $(\nu \times \eta) \perp (\mu \times \eta)$.

20.16. **Problem 16.** [Remus Nicoara] Let $\{\mu_n\}_{n=1}^{\infty}$ be finite measures on $(X, \mathcal{M})$ such that $\mu = \sum_{n=1}^{\infty} \mu_n$ is also a finite measure. If $\nu \perp \mu_n$ for all n, show that $\nu \perp \mu$.

21. Normal Families

21.1. **Problem 1.** a. [MathStackExchange] Define the set

$$\{f : \mathbb{D} \to \mathbb{C}, f(z) = \sum_{n=0}^{\infty} a_n z^n, |a_n| \leq n^2\}$$

and show it is a normal family.

b. Why does the argument used in part a fail if we extend the family f to a larger disc, say $B(0, 2)$?

21.2. **Problem 2.** Let $A := A(0, r_1, r_2)$ be an annulus in $\mathbb{C}$ centered at 0, with $0 < r_1 < r_2$. Show that

$$\mathcal{F} := \{f \text{ analytic}, f : A \to \mathbb{C}, |f(z)| \leq \frac{1}{|z|}, z \in A\}$$

is a normal family.

21.3. **Problem 3.** [Carl Sundberg] Let G denote the upper half-plane in $\mathbb{C}$. Let f be a function such that $f \in H(G)$, $|f(z)| \leq 1$ for all $z \in G$, with $f(iz) \to 0$. Prove that $f(z) \to 0$ non-tangentially as $z \to 0$. That is, show any sequence $\{z_n\}_{n=1}^{\infty} \subset G$ such that $z_n \to 0$ is such that $f(z_n) \to 0$ as $n \to \infty$.

21.4. **Problem 4.** [Carl Sundberg] Let $G \subset \mathbb{C}$ be an open domain and define

$$\mathcal{F} := \{f \in H(G), \text{univalent}, f(G) \subset \mathbb{D}, f(z_0) = 0\}$$

where $z_0 \in G$ is fixed. Show that $\mathcal{F}$ is normal.

22. Parabolic PDE

22.1. **Problem 1.** Fix $T > 0$. Suppose $\phi_L, \phi_R : (0, T) \to \mathbb{R}$ and $\psi : [0, 1] \to \mathbb{R}$ are smooth functions. Prove that the following IBVP has at most one solution.

$$\begin{cases} u_{xt} - u_{xxx} = 0, (x, t) \in (0, 1) \times (0, T) \\ u(0, t) = \phi_L(t), t \in (0, T) \\ u(1, t) = \phi_R(t), t \in (0, T) \\ u(x, 0) = \psi(x), x \in (0, 1) \end{cases} .$$

22.2. **Problem 2.** Fix $T > 0$. Prove that the zero function is the only solution to this IBVP.

$$\begin{cases} u_t^3 + u + 1 = 0, (x,t) \in (0,1) \times (0,T) \\ u(0,t) = 0, t \in (0,T) \\ u(1,t) = 0, t \in (0,T) \\ u(x,0) = 0, x \in (0,1) \end{cases}.$$

22.3. **Problem 3.** a. Let $u \in C^2(\mathbb{R} \times (0,\infty))$ solve $u_t - u_{xx} = 0$. Prove that the scaled solution $u_\lambda(x,t) = u(\lambda x, \lambda^2 t)$ also solves this PDE.

b. Let $u \in C^4(\mathbb{R} \times (0,\infty))$ solve $u_{tt} - u_{xxxx} = 0$. Prove that the scaled solution $u_\lambda(x,t) = u(\lambda x, \lambda^2 t)$ also solves this PDE.

c. Let $u \in C^2(\mathbb{R} \times (0,\infty))$ solve $u_t - u_{xx} = u$. Show scaling the solution $u_\lambda(x,t) = u(\lambda x, \lambda^2 t)$ does not automatically guarantee that u_λ solves the same PDE.

d. Let $u \in C^2(\mathbb{R} \times (0,\infty))$ solve $u_t - u_{xx} = uu_t$. For what algebraic relationship between α, β is it guaranteed that $u_\lambda(x,t) = u(\lambda^\alpha x, \lambda^\beta t)$ also solves the same PDE?

23. Power Series and Laurent Series

23.1. **Problem 1.** a. [UTK Analysis Prelim August 2011 Problem 4] Prove that

$$\sum_{n=1}^{\infty} \frac{1}{(z-n)^2}$$

defines an analytic function in $\mathbb{C} \setminus \mathbb{N}$.

b. [My extension] Prove that

$$\sum_{n=1}^{\infty} \frac{-2}{(z-n)^3}$$

defines an analytic function in $\mathbb{C} \setminus \mathbb{N}$.

23.2. **Problem 2.** a. [UTK Analysis Prelim August 2012 Problem 2] Let $f(z) = \frac{\cos(2z)}{\log(1+3z)}$. Determine the nature of the singularity that f has at 0 and find its singular part at 0. Here are two ways to try the problem. Do both.

i) Expand the numerator and denominator into power series.

ii) Use the complex version of L'Hopital's Rule to determine the limiting behavior of f near the singularity.

b. [UTK Analysis Prelim August 2012 Problem 2] Consider the Laurent series of f at 0. Determine the largest set where this series converges.

c. [My variant] Use part b to find an upper bound on the radius of convergences for the Laurent series of

$$g(z) := \frac{\sin(4z)}{\log(1+3z)}$$

centered at 0.

23.3. **Problem 3.** Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series convergent in $B(0,1)$, and let $\{M_n\}_{n=0}^{\infty} \subset \mathbb{C}$ be a sequence of complex numbers such that $\sum_{n=0}^{\infty}|M_n|< \infty$. If $g(z) = \sum_{n=0}^{\infty} a_n M_n z^n$, prove the radius of convergence of this power series centered at 0 is at least 1.

23.4. **Problem 4.** a. Determine the radius of convergence from the Laurent Series

$$f(z) = \sum_{n=0}^{\infty} \frac{n^2}{n^2+1} z^n + \frac{1}{\log(3z+2)^5}$$

centered at 0.
b. Determine the smallest $\alpha > 0$ such that the radius of convergence of

$$g(z) = \sum_{n=0}^{\infty} \frac{n^{2\alpha}}{n^2+1} z^n + \frac{1}{\log(3z+2)^5}$$

is smaller than the radius of convergence of f.

24. Separation of Variables

24.1. **Problem 1.** I was once trying to investigate solutions of the Korteweg-de-Vries (KdV) equation

$$u_t + uu_x + u_{xxx} = 0$$

by first looking at solutions to the Inviscid Burgers' Equation

$$u_t + uu_x = 0.$$

I realized that if $u_{xxx} = 0$ for a solution to Inviscid Burgers' Equation, then that function u also solves the KdV Equation. As a result I tried looking for special separable solutions to this IBVP for the Inviscid Burgers Equation

$$\begin{cases} u_t + uu_x = 0, (x,t) \in (0,\pi) \times (0,\infty) \\ u(0,t) = 0, t \in (0,T) \\ u(\pi,t) = 0, t \in (0,T) \\ u(x,0) = u_0(x), x \in (0,1) \end{cases}$$

where $u_0 \in L^2((0,\pi))$. For this particular problem, consider separable solutions of the form

$$u(x,t) = a(t)(c_1x^2 + c_2x + c_3),$$

where $a \in C^1([0,\infty))$ and $c_1, c_2, c_3 \in \mathbb{R}$ with $c_1 \neq 0$. By mimicking the classical separation of variables technique, show that the only such solutions are constant functions.

24.2. **Problem 2.** The *beam equation* in $\mathbb{R} \times \{t > 0\}$ is

$$u_{xxxx} + u_{tt} = 0$$

Use separation of variables to derive each of the following solutions to the PDE (no initial or boundary data). For each solution, which eigenvalue identifies it?

$$u(x,t) = \sin(2x)\cos(4t)$$
$$u(x,t) = \sin(2x)\sin(4t)$$
$$u(x,t) = \cos(2x)\cos(4t)$$
$$u(x,t) = \cos(2x)\sin(4t)$$

24.3. **Problem 3.** Use separation of variables to show that for any eigenvalue $\lambda \in \mathbb{R}$ the function

$$u(x_1, x_2) = e^{-x_1 + x_2(\lambda - 1)} \sin(\lambda x_1)$$

is a solution to the nonlinear PDE

$$2u + 2u_{x_1} + 2u_{x_2} + u_{x_1x_1} + u_{x_2x_2} = 0$$

in all of $\mathbb{R}^2$. Do not worry about initial or boundary data for this problem, hence we don't need to restrict eigenvalues (though we get a trivial solution if $\lambda = 0$).

24.4. **Problem 4.** [UTK PDE Prelim January 2018 Problem 6] Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain, $T > 0$, and let $a \in \mathbb{R}^n$ be a vector. Suppose $u \in C^2(\bar{\Omega} \times [0,T])$ solves

$$\begin{cases} u_t = \triangle u + a \cdot \bigtriangledown u + u^2, (x,t) \in \Omega \times (0,T] \\ u = 0, \qquad (x,t) \in \partial\Omega \times (0,T] \\ u = 0, \qquad (x,t) \in \Omega \times \{t = 0\} \end{cases}$$

Show that $u \geq 0$ on $\Omega \times (0,T]$ in two ways:
a. Replicate the proof of the maximum principle for the heat equation
b. Let $v = ue^t$ and form a PDE for v. Then use a maximum principle for parabolic PDE.
c. Now show that $u_t \geq 0$ on $\Omega \times (0,T]$.
d. [My variant] For which vectors $a \in \mathbb{R}^n$ is a solution to this PDE necessarily subharmonic?

25. Residues

25.1. **Problem 1.** Let $\Omega := \mathbb{C} \setminus [-1,1]$ and let γ be a closed curve in Ω.
a. [UTK Analysis Prelim August 2011 Problem 6a] If $f(z) = \frac{2}{z^2-1}$, show that $\int_\gamma f(z)dz = 0$.
b. [My extension] If $f(z) = \frac{2}{(z+1)^2(z-1)}$, show that $\int_\gamma f(z)dz = 0$.

26. Sigma Algebras and Set Theory

26.1. **Problem 1.** a. Let $\mathcal{A}_1, \mathcal{A}_2$ be algebras on a set X. Show that $\mathcal{A}_1 \cap \mathcal{A}_2$ is an algebra.

b. Let $\mathcal{A}_1, \mathcal{A}_2$ be σ-algebras on a set X. Show that $\mathcal{A}_1 \cap \mathcal{A}_2$ is a σ-algebra.

c. [MATH 545 Fall 2019 Homework problem] Find $\mathcal{A}_1, \mathcal{A}_2$ σ-algebras on a set X such that $\mathcal{A}_1 \cup \mathcal{A}_2$ is NOT a σ-algebra on X.

d. Let $\mathcal{A}_1, \mathcal{A}_2$ be σ-algebras on a set X. Show that the symmetric difference of $\mathcal{A}_1$ and $\mathcal{A}_2$ will NEVER be a σ-algebra.

e. Show that the collection $\mathcal{C} := \{(-\infty, b) : b \in \mathbb{R}\}$ generates the Borel σ-algebra $\mathcal{B}$ on $\mathbb{R}$.

26.2. **Problem 2.** a. Suppose $\{\mathcal{M}_i\}_{i=1}^{\infty}$ is a collection of monotone classes on a set X. Prove that $\mathcal{M} := \cap_{i=1}^{\infty} \mathcal{M}_i$ is also a monotone class on X.

b. Suppose $\{\mathcal{M}_i\}_{i=1}^{N}$ is a finite collection of monotone classes on a set X. Prove that $\mathcal{M} := \cup_{i=1}^{N} \mathcal{M}_i$ is also a monotone class on X.

c. Prove by means of a counterexample that the claim in part b no longer holds if the union is countably infinite.

26.3. **Problem 3.** a. Let $A \subset [0,1]$ be Borel measurable. Then show for any $\epsilon > 0$ there exists a countable collection of disjoint open intervals $H_i := (c_i, d_i)$ for which $m\left(\cup_{i=1}^{\infty} H_i \setminus A\right) < \epsilon$. What is $\lim_{i\to\infty} m(H_i)$?

b. Let $A \subset [0,1]$ be Borel measurable. Show for any $\epsilon > 0$ there exists U open and K compact such that $K \subset A \subset U$ and $m(U \setminus K) < \epsilon$.

26.4. **Problem 4.** Let $(X, \mathcal{A})$ be a measurable space, and let $A_1, A_2, \ldots \in \mathcal{A}$. Show that if B is the collection of $x \in X$ belonging to infinitely many of $A_1, A_2, \ldots$, then

$$B = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

is in $\mathcal{A}$.

26.5. **Problem 5.** [Remus Nicoara] Consider the collection of half-open intervals

$$\left(\frac{1}{N+1}, \frac{1}{N}\right]$$

for $N \in \mathbb{N}^+$. Explicitly state the σ-algebra generated by these sets.

27. Simply Connected Domains

27.1. **Problem 1.** a. [This exercise comes from [Sar]] Let $\{G_n\}_{n=1}^{\infty}$ be a decreasing sequence of simply connected domains such that the interior of $\cap_{n=1}^{\infty} G_n$ is nonempty. Prove that every connected component of the interior of $\cap_{n=1}^{\infty} G_n$ is simply connected.

b. Find a selection $\{G_n\}_{n=1}^{\infty}$ a decreasing sequence of simply connected domains where $\cap_{n=1}^{\infty} G_n$ is a singleton and verify that singletons are simply connected.

27.2. **Problem 2.** Suppose G is a simply connected domain in $\mathbb{C}$. Prove that for any $\theta \in [0, 2\pi]$, $H := e^{i\theta} G$ is also simply connected.

27.3. **Problem 3.** Let E denote the double quarter plane $E := \{\Re z > 0, \Im z > 0\} \cup \{\Re z < 0, \Im z < 0\}$

a. Demonstrate that E is not simply connected.

b. Why is E not conformally equivalent to the unit disc $\mathbb{D}$?

27.4. **Problem 4.** Suppose that G is an open, bounded domain in $\mathbb{C}$ such that $\mathbb{D} \subset G$. Suppose f is analytic in G. Then prove that there exist $E, F : G \to \mathbb{C}$ analytic, and ϕ that analytically and conformally maps G into $\mathbb{D}$ such that

$$\frac{d^2}{dz^2}(E \circ \phi(z)) = \phi''(z)F(\phi(z)) + (\phi'(z))^2 f(\phi(z))$$

for all $z \in G$.

28. Hints: Derivative Inequalities for Holomorphic Functions and the Identity Theorem

Problem 1: use the Cauchy Integral formula on a disc with large radius.
Problem 4a: use the Identity Theorem for holomorphic functions.
Problem 7a: break into cases: $m_1 = m_2, m_1 < m_2, m_1 > m_2$.
Problem 11: notice $u(0) = v(0)$ and use the mean-value property for v.
Problem 12b: let $\phi(z) := \frac{\frac{1}{2}-z}{1-\frac{1}{2}z}$. Juxtapose ϕ and f so you can apply Schwarz Lemma.

29. Hints: Differentiation in the Lebesgue Sense

Problem 2d: Notice that $fg = \frac{1}{2}[(f+g)^2 - f^2 - g^2]$.

30. Hints: Elliptic PDE

Problem 8: Write $\triangle u + u^3 - u^4$ as $\triangle u + cu$ for a nonpositive continuous function c. Consider carefully the conditions necessary to invoke the Hopf Lemma.

31. Hints: Hyperbolic PDE

Problem 1: for part b, integrate the inequality derived in part a.

Problem 2c-d: first prove the bound $t^2 - |y-x|^2 \geq t|y-x|$. In part b it then follows that $t^2 - |y-x|^2 \geq t|y-x|^2$.

Problem 2f: the integral over the boundary and the $y-x$ factor that arise inside the integrand suggest using the Divergence Theorem, but you will need to rescale the vector $y-x$ first to ensure it's of unit length.

Problem 4a: use Duhamel's Formula to get a specific formula for u, and use that as the basis for estimates. You'll need a certain inequality to introduce the L^p norm of g.

Problem 12a: if we can make the domain of integration independent of time, we can differentiate under the integral sign. How can we make the domain of integration independent of time?

Problem 15c: find a formula for $E''(t)$ and then use to estimate $3E'(t)^2E''(t)$.

Problem 16: for part a, use Kirchhoff's Formula to get an integral formula with g, and then use Cauchy-Schwarz to turn this into a g^2; for part b, use some of the intermediate steps from part a to get a formula for $u_t(0,t)$, then calculate a bound for $2u(0,t)u_t(0,t)$.

32. Hints: Integrals (Complex Valued)

Problem 2: prove that all convex sets are simply connected as a first step.

33. Hints: Integrals and Limit Theorems

Problem 4a: prove the claim first in the case where f is a step function.

34. Hints: Interpolation Inequalities

Problem 1a, b: assume u and v satisfy the given inequality and use a series of change of variables steps to get a similar inequality for u_λ and v_λ. The resulting inequality must be independent of λ.

Problem 5: Use Poincaré's Inequality to estimate the greater side of the inequality from below.

35. Hints: Laplace's Equation and Harmonic Functions

Problem 1: for parts a and b use Hölder's Inequality. For part c use the Mean-Value Property.

Problem 2: For parts a and b, notice that $\int_\Omega u \triangle u dx = 0$. For part c, notice $\int_\Omega f(u) \triangle u dx = 0$ for any $f \in C^0(\Omega)$. Pick a suitable f.

Problem 3: For part c, look to use part a. For part d, let $T(r) = u(x)$ and $U(r) = \frac{(r+|x|)r^{n-2}}{(r-|x|)^{n-1}} u(0)$ for a fixed $x \in B(0,r)$ and $0 \leq r \leq R$. Calculate $T'(r)$ and $U'(r)$.

Problem 6: For part a, look to use the Monotone Convergence Theorem. For part b, look to use the Dominated Convergence Theorem.

Problem 10: take x to be fixed. What PDE and boundary condition does h satisfy when x is fixed?

36. Hints: Parabolic PDE

Problem 1: Let v be the difference of two solutions to the IBVP and let $E(t) := \int_0^1 v_{xx}^2(x,t)dt$.

37. Hints: Separation of Variables

Problem 3: Separation of variables is generally for linear PDEs but the one we're given is nonlinear. Consider using the substitution $s(x_1,x_2) = e^{x_1+x_2}u(x_1,x_2)$ to rewrite the PDE and the proposed solution.

38. Hints: Sigma-Algebras and Set Theory

Problem 2c: consider your proof of part b and where it may break down in the infinite case.

39. Hints: Simply Connected Domains

Problem 4: show that the set G is simply connected. This will indicate what role the function ϕ will play.

REFERENCES

[Ev] L.C. Evans, *Partial Differential Equations*, 2nd Edition, American Mathematical Society, 2013.
[Han] Q. Han, *A Basic Course in Partial Differential Equations*, 1st Edition, Americal Mathematical Society, 2011.
[Sar] D. Sarason, “Complex Function Theory," 2nd edition (2007). American Mathematical Society.