WHY TO USE CONTINUOUS FE FUNCTIONS OVER DISCONTINUOUS ONES

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1. MOTIVATION

Finite element methods for solving partial differential equations often like to use continuous piecewise polynomial functions (especially linear ones) to approximate solutions. For various reasons that are problem and context-dependent, using continuous piecewise linear functions is often favorable to using discontinuous ones. This may seem counter-intuitive at first: one may think it is easier to approximate some random function with discontinuous polynomials because there are a lot more of them, loosely speaking. As it turns out, this is often not the case because the continuous variants belong to Sobolev Spaces, whereas the discontinuous variants typically do not.

In this note we assume $\Omega \subset \mathbb{R}^n$ is a bounded, open set, $p \in (1, \infty)$.

Definition 1.1 (Weak Derivative)**.** *A function $w \in L^p(\Omega)$ is said to have **weak derivative** v if for all $\varphi \in C_0^\infty(\Omega)$ and $i \in \{1, 2, \dots, n\}$, the integration by parts identity holds:*

$$\int_\Omega w(x)\partial_{x_i}\varphi(x)dx \;=\; -\int_\Omega v(x)\varphi(x)dx. \tag{1.1}$$

2. CASE STUDY 1: CONTINUOUS PIECEWISE

As a prototype for continuous piecewise linear finite element methods, we define the function

$$\phi_1(x) \;=\; \begin{cases} 1, 0 \leq x \leq \dfrac{1}{2} \\ x + \dfrac{1}{2}, \dfrac{1}{2} \leq x \leq 1 \end{cases} \tag{2.1}$$

on $(0, 1)$ and show that this function has weak derivative

$$v_1(x) \;=\; \begin{cases} 0, 0 \leq x \leq \dfrac{1}{2} \\ 1, \dfrac{1}{2} \leq x \leq 1 \end{cases} \tag{2.2}$$

Note that in general, to prove a function is a weak derivative of another, you must have a "candidate" in mind and check it with the definition; the definition does not provide intuition on what the weak derivative could/should be.

In any case, we proceed as follows: let $\varphi \in C_0^\infty(0, 1)$ be an arbitrary smooth function that, importantly, vanishes on the endpoints, and we show that the integration by parts formula is true

Date: January 26, 2024.

by using the formulas for ϕ_1 and v_1. Since we have a piecewise function, break up the integration based on where the “pieces" are defined.

$$\int_0^1 \phi_1(x)\varphi'(x)dx = \int^{\frac{1}{2}} \phi_1(x)\varphi'(x)dx + \int_{\frac{1}{2}}^1 \phi_2(x)\varphi'(x)dx = \int_0^{\frac{1}{2}} \varphi'(x)dx + \int_{\frac{1}{2}}^1 \left(x+\frac{1}{2}\right)\varphi'(x)dx$$

Since φ is smooth, we can use the Fundamental Theorem of Calculus on the first integral and integration by parts on the second to get that the above equals

$$\varphi\left(\frac{1}{2}\right) - \varphi(0) - \int_{\frac{1}{2}}^1 \varphi(x)dx + \left[\left(x+\frac{1}{2}\right)\varphi(x)\right]_{\frac{1}{2}}^1$$

Since φ vanishes on the endpoints the above simplifies to $-\int_{\frac{1}{2}}^1 \varphi(x)dx$, regardless of the real value of the function φ. Now, due to the definition of v_1, this expression actually equals $-\int_0^1 v_1(x)\varphi(x)dx$, which means v_1 is the weak derivative of ϕ_1!

3. Case Study 2:

Now we show with a second example where continuity of ϕ_1 really came into play, by showing another example. Consider the function

$$\phi_2(x) \;=\; \begin{cases} -1, 0 \le x \le \dfrac{1}{2} \\ 1, \dfrac{1}{2} \le x \le 1 \end{cases} \tag{3.1}$$

One would hope that this function has a weak derivative equal to 0 everywhere, but the discontinuity causes a problem there. To make this explanation a bit more rigorous, we follow the official definition. Suppose, for sake of contradiction, that there exists a function v_2 (say, in $L^1(0,1)$) for which

$$\int_0^1 \phi_2(x)\varphi'(x)dx \;=\; -\int_0^1 v_2(x)\varphi(x)dx \tag{3.2}$$

for any $\varphi \in C_0^\infty(0,1)$. Using the formula for ϕ_2 gives

$$\int_0^1 \phi_2(x)\varphi'(x)dx = -\int_0^{\frac{1}{2}} \varphi'(x)dx + \int_{\frac{1}{2}}^1 \varphi'(x)dx$$

Now we can use the Fundamental Theorem of Calculus, and terms cancel:

$$-\int_0^{\frac{1}{2}} \varphi'(x)dx + \int_{\frac{1}{2}}^1 \varphi'(x)dx = -\varphi\left(\frac{1}{2}\right) + \varphi(0) + \varphi(1) - \varphi\left(\frac{1}{2}\right) = -2\varphi\left(\frac{1}{2}\right)$$

By no means can we guarantee that this point value always equals the integral $-\int_0^1 v_2(x)\varphi(x)dx$. In particular, we cannot make the value of φ at a point appear from this integral expression.

This is the typical behavior of a discontinuous piecewise polynomial that arises in Discontinuous Galerkin methods. Of course, there are times where this is the necessary or even preferred means of solving a problem.

Bibliographic Note: For a good first introduction to weak derivatives, I personally recommend looking at Chapter 5 of [1].

REFERENCES

[1] L.C. Evans. *Partial Differential Equations Second Edition*. American Mathematical Society, 2010.